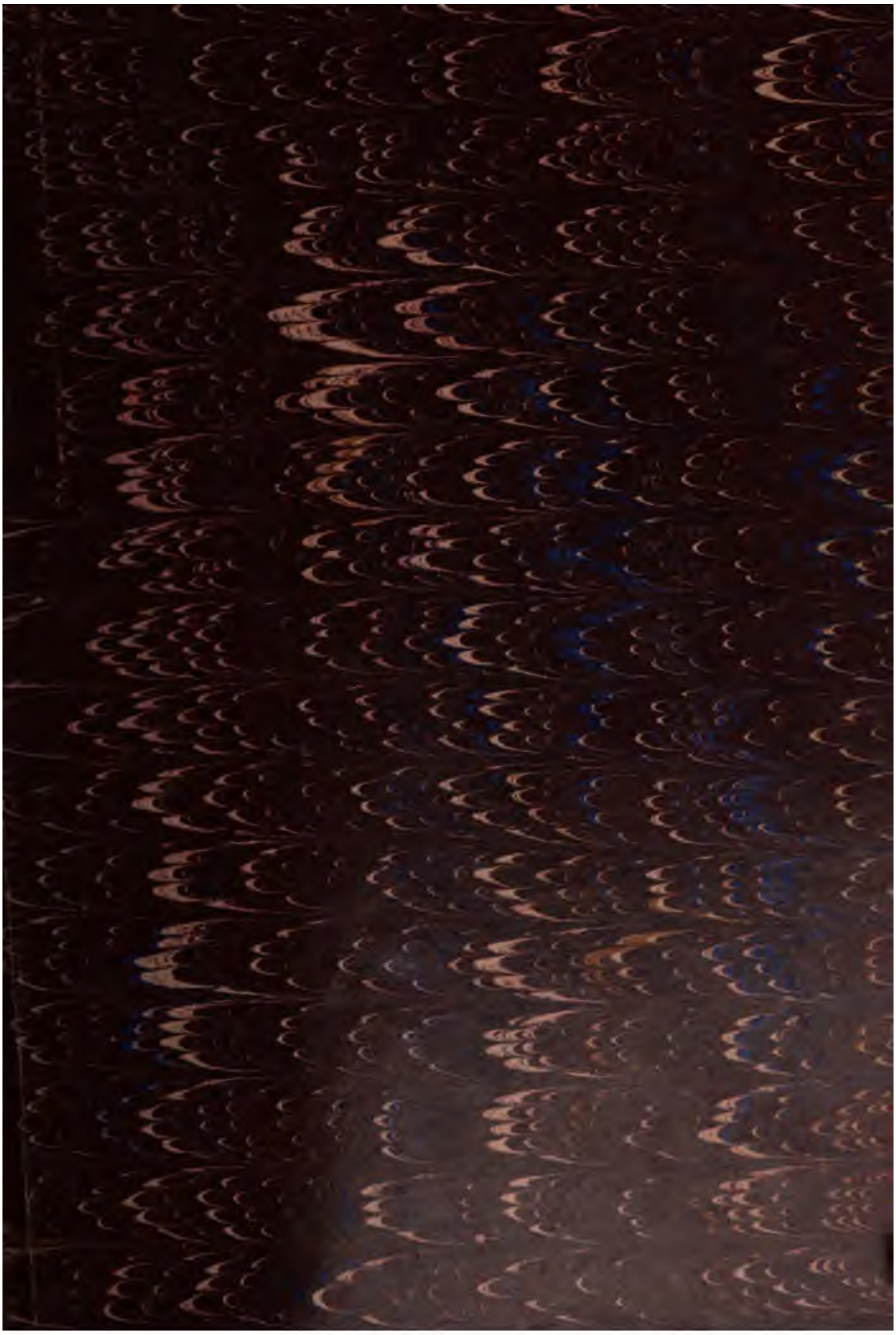

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THE ALTERNATE CURRENT TRANSFORMER

IN THEORY AND PRACTICE.

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VOLUME I.

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PREFACE.

THE present treatise is an attempt to place before the reader an elementary account of the principles which underlie the operations and the use of the Alternating-Current Transformer. It frequently happens that whilst practical students are in possession of clear ideas on the fundamental phenomena exhibited in the application and generation of continuous or steady electric currents, the endeavour to cope with similar problems concerning periodic currents finds them in want of some special assistance to enable them to deal with the peculiar difficulties which surround such study. Particularly is this the case now that periodic or alternating currents are largely employed in electric illumination, and the necessity arises for all students of electro-technics to be prepared to deal with and appreciate the particular questions which thus arise. The practical employment of periodic currents and their inductive transformation is becoming so important that it seemed probable service would be rendered to those dealing with these matters by placing together the main outlines of the theory and of the applications of electric-current induction. The work here presented is an attempt, no doubt very imperfect, to realise this aim. In the first volume the General Phenomena and Effects of Electric-Current Induction, Periodic Currents, and Electro-Magnetic Induction are considered, together with so much collateral matter as is necessary to a clear comprehension of the subject. In the second volume it is proposed to consider the subjects of Practical Measurements, the Construction, Design, and use of Induction Transformers, and the applications in Lighting, Welding, and other Technical Work.

The subject of alternate-current generators is not included in this scheme, partly because it has to a considerable extent been dealt with in existing treatises, but chiefly because the discussion of the inductive transformation of electric currents affords matter more than sufficient to occupy the limits of an elementary treatise. In dealing with the subject the desire of the author has been to collect out of the technical journals and the published transactions of Societies the contributions of various writers who have especially added matter to our knowledge in this department having a practical bearing, and to place it together in such a form as to be more easily assimilated by students who have not the facilities to do this for themselves. The study and employment of periodic electric currents have forced the attention both of experimentalists and of theorists to electrical phenomena, which are in themselves of the highest degree of importance, and this attention has been rewarded by a marked advance in the knowledge of learners of all ranks. In his introductory address to the British Association at Montreal, Lord Rayleigh remarked that "the introduction of powerful alternate-current machines by Siemens, Gordon, Ferranti, and others is likely to have a salutary effect in educating practical electricians, whose ideas do not rise easily above ohms and volts." It has long been known that, when the changes are sufficiently rapid, the electric current phenomena are governed more by electric inertia than by mere resistance, and the introduction of this inertia factor alters in a striking manner the whole aspect of phenomena; but it is only quite recently that attention has in a very marked degree been directed to these effects, with the result that many conventional modes of regarding electrical phenomena are retiring into the background, and a vista of new and altogether more ample knowledge is opening up.

The pages of this book will bear evidence in some very slight degree of the extent to which this is the case and of the names of those to whom it is due. At the head of this long line of illustrious investigators stand the pre-eminent names of Faraday and Henry. On the foundation-stones of truth laid down by them all subsequent builders have been content to rest. The "Electrical Researches" of the one have been the guide of the experimentalist no less than the instructor of the student, since their orderly and detailed statement, alike of triumphant discovery and of suggestive failure, make them independent of any commentator. The "Scientific Writings" of Henry* deserve hardly less careful study, for in them we have not only the lucid explanations of the discoverer, but the suggestions and ideas of a most profound and inventive mind, and which indicate that Henry had early touched levels of discovery only just recently becoming fully worked. Latest amongst these workers may be enrolled the name of Dr. Hertz, and the author takes this opportunity of expressing to Mr. De

* "The Scientific Writings of Prof. Joseph Henry." Washington: 1886. Published by the Smithsonian Institution.

Tunzelmann his thanks for permission to appropriate *en bloc* the excellent account of Hertz's researches, from his pen, which appeared in the pages of THE ELECTRICIAN. This forms by itself one section in Chapter V., and will it is hoped be sufficient to enable this valuable work to be appreciated. It is evident without remark that the limits of an elementary treatise prevent anything of the character of exhaustive treatment in any of the subjects which are touched upon, even if this was possible in the case of a science exhibiting such a rapid rate of growth as that of Electro-Magnetism. Students having the necessary acquirements will naturally seek for this more complete treatment in the writings of well-known mathematical physicists. The pages of back volumes of THE ELECTRICIAN are a mine in which the diligent reader can delve for a part of this more copious treatment; and it is perhaps to be regretted that the valuable contributions of Mr. Oliver Heaviside to the electrical branch of theoretical science are not yet classified and condensed into a form more accessible and handy than the volumes of a journal.

For others, content to possess themselves of a more elementary knowledge of the chief phenomena of electro-magnetic induction, the following pages may serve as a guide.

Some portions of the book have already appeared as contributions to THE ELECTRICIAN, but these have been extended and carefully revised before being again here presented to the reader.

J. A. F.

London, 1889.

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CHAPTER I.

INTRODUCTORY.

§ 1. **Introductory.**—The 29th day of August, 1831, was an important day in the calendar of electrical discovery. On that day Faraday, in the prime of his powers, set himself to commence his "Electrical Researches." Looking on the group of electric phenomena with an eye eager to see physical analogies, and confident that where these exist they may prove suggestive for further research, he asked himself, in the first place, if it were possible there was any effect in the case of electric currents analogous to that known as electrostatic induction. An insulated conductor possessing an electric charge when introduced into a closed chamber having conducting walls calls forth upon them an equal charge of an opposite sign. This *induced electrification* is invariably present, no matter how far off the walls of the enclosing chamber may be, and all surrounding conductors share in the duty of carrying a portion of the induced charge. At a later date, when Faraday viewed this phenomenon of electrostatic induction by the aid of the education he had received in dealing with magnetic lines of force, he was able to imagine certain lines of electrostatic force proceeding in all directions from the surface of a charged body. Wherever they terminated, whether on neighbouring conductors or on walls of an enclosing chamber, they developed on these "corresponding points" a charge equal and opposite to that of the surface at the point from which they took their rise. Eleven years previously Prof. H. C. Œrsted had made Copenhagen famous as the birth-place of the discovery that an electric current passing through a metallic wire magnetises it circularly and creates round it a magnetic field, the direction of the lines of magnetic force being closed curves surrounding the axis of the wire. Faraday placed these two phenomena side by side before his mental vision, and he asked himself whether it was possible that the magnetic field of force generated round a current-carrying conductor could develop in an adjacent circuit an *induced current* just as the charged body calls forth an *induced electrostatic charge* on the neighbouring conductors.

In 1825, in the month of November, Faraday stretched alongside of a wire connected with a galvanometer another through which an electric current was flowing, but both then and on December 2nd, 1825, and on April 22nd, 1828, he had to record of his experiment that it gave "no result." A very little step in experimental research often separates failure from success. A reversal of operations, a change of some dimension, an alteration of some proportion, is often all that is needed to step from the region of failure into the field of discovery and achievement. In this case it was the apparently trivial one of starting the electric current in one wire *before* completing the circuit of the galvanometer.

The one thing, it seemed, that this preliminary work did disprove was the notion that a continuous steady current in one conductor could generate a continuous current in another adjacent conductor relatively at rest to the first. It is possible that some conception of the above nature had been dominant in the mind of Faraday before these trials had convinced him that the effect, if existing at all, was not detectable with his apparatus.* We cannot describe the experimental results of the autumn months of 1831 better than they have been given in Faraday's own words in the laboratory note-books of the Royal Institution.† His first experiment, detailed in the second paragraph, records the epoch-making discovery by which he will be for ever known. He wrote: "I have had an iron ring made (soft iron), iron round and $\frac{7}{8}$in. thick, and ring

* It does not follow that continuous-current induction is absolutely untrue. It might be possible to realise it with enormously strong inducing currents and very delicate galvanometric methods.

† *See* Dr. Bence Jones's "Life of Faraday," Vol. II., p. 2.

6in. in external diameter. Wound many coils of copper round one-half of it, the coils being separated by twine and calico; there were three lengths of wire, each about 24ft. long, and they could be connected as one length or used as separate lengths. By trials with a trough, each was insulated from the other. Will call this side of the ring A. On the other side, but separated by an interval, was wound wire in two pieces, together amounting to about 60ft. in length, the direction being as with former coils. This side call B. Charged a battery of ten pairs of plates 4in. square. Made the coil B side one coil, and connected its extremities by a copper wire passing to a distance and just over a magnetic needle (3ft. from wire ring), then connected the ends of one of the pieces on A side with battery; immediately a sensible effect upon needle. It oscillated, and settled at last in original position. On breaking connection of A side with battery, again a disturbance of the needle."

On September 24th he resumed his attack. He prepared an iron cylinder and wound on it a helix of insulated wire. The ends of the helix were connected with a galvanometer. The iron was then placed between the poles of bar magnets. Every time the magnet poles were brought in contact with the ends of the iron cylinder the galvanometer indicated a current, the effect being, as in former cases, not permanent, but a mere momentary push or pull.

But the full meaning of this hardly appeared clear, and on October 1st he once more laid siege to the fortress. Preparing a battery of 100 pairs of plates, each 4in. square, and charged with a mixture of nitric and sulphuric acids, he arranged to send the current from this through a wire of copper 203ft. long wound round a block of wood. Round the same block, and wound parallel to the first, was a second wire, of equal length to the first, insulated from it. This second wire he joined up to the terminals of his galvanometer, and then when the battery connection was *made* or *broken* with the first wire he noticed a small but sudden jerk of the needle, one way when the current was made, the other way when it was broken. The clue to the real phenomenon was now in his hand, and guided by it he stepped over a series of confirmatory experiments, and entered as a triumphant conqueror into the stronghold wherein the whole truth lay hid.

Writing on November 29th to his friend, Mr. R. Phillips, he says :—" Now, the pith of all this I must give you very briefly. When an electric current is passed through one of two parallel wires it causes at first a current in the same direction through the other, but this induced current does not last a moment, notwithstanding the inducing current (from the voltaic battery) is continued. All seems unchanged except that the principal current continues its course. But when the current is stopped, then a return current occurs in the wire under induction of about the same intensity and momentary duration, but in the opposite direction to that first formed. Electricity in currents, therefore, exerts an inductive action like ordinary electricity, but subject to peculiar laws. The effects are a current in the same direction when the induction is established, a reverse current when the induction ceases, and a *peculiar state* in the interim."

The path for valuable discovery now lay open. Fully familiar with the work of Ampère and Arago, Faraday knew that a closed circuit conveying an electric current affects all surrounding space with magnetic force, and that in particular a small closed circular current can, as far as magnetic action is concerned, be exactly replaced by a very thin disc of steel, whose edge coincides with the line of the closed current, and which is magnetised everywhere in a direction perpendicular to its surface. Such a normally magnetised disc is called a *magnetic shell*. It follows that a helix of wire, which may be regarded as a number of closely approximate circular currents nearly in the same plane, would be magnetically equivalent to a number of magnetic shells piled one above the other, with similar polar faces turned the same way. But such an arrangement of shells would form a cylindrical magnet, and therefore a helix of wire or solenoid in which a current is flowing is for all external space the magnetic equivalent of a cylinder of steel of the same dimensions magnetised uniformly in a longitudinal direction. It remained, therefore, to test this hypothesis.

The fifth day of his experiments was October 17th, and on that day he thus notes in the laboratory book the results:—"A cylindrical bar magnet $\frac{3}{4}$in. in diameter and $8\frac{1}{2}$in. in length had one end just inserted into the end of a helix of wire 220ft. long. It was then quickly thrust in the whole length and the galvanometer needle moved; then pulled out again, and again the needle moved, but in the opposite direction. This effect was repeated every time the magnet was put in or out, and therefore a wave of electricity was so produced from mere approximation of a magnet."

Exactly twenty years afterwards, in the 28th and 29th series of his "Researches," Faraday illuminated, by the exactness and clearness of his experimental method, the whole behaviour of magnets towards closed conducting circuits. It is probable that even at this time he had learned to think of a magnet as carrying with it, as part of itself, a whole system of lines of magnetic force, which emanate from it and surround it. The system of lines of force moves with the magnet wherever it goes. Regarding the production of a current in the helix by a magnet thrust into it, Faraday pictured to himself the advancing magnet as pushing its lines of magnetic force across the coils of wire of the helix, and "cutting" or intersecting them in its progress towards its final position in the coil. The conclusion to which he was led by this reflection seemed to be that the very essence of the effect was the movement across one another of a line of force and a portion of a conducting circuit. If this was so, then the result could be obtained by a more simple and obvious method. The ninth day, October 28th, saw these ideas put to further crucial test. Taking the great permanent horse-shoe magnet of the Royal Society, he placed a copper disc so that it was free to revolve on an axis placed in the line of the poles. Soft iron pole pieces were then adjusted to create a powerful magnetic field, the lines of force of which passed through the disc at right angles to its surface. The wires of the galvanometer were made to press against the disc, one near the axis, and the other near the edge. When the disc remained stationary, no current whatever was manifested, but on causing the disc to revolve on its axis a permanent and steady current traversed the galvanometer. This experiment was conclusive. The operation taking place during the revolution of the disc could be viewed as consisting simply in the continual movement of any radial section of the disc across a stream of lines of magnetic force flowing at right angles to its surface. The continuous current resulted from the fact that the motion of that radial section of the disc was always the same relatively to the stream of force. On November 4th Faraday reduced the conception to its utmost simplicity. Taking in his hand the mere closed galvanometer wires he passed a portion of the loop between the poles of his large permanent magnet in such a way that the direction of that part of the loop between the poles was at right angles to the direction of the magnetic force, and the direction of the movement was at right angles to the direction of the force and that portion of the conductor. The galvanometer deflected and showed the presence of a momentary current *at* the instant when the intersection took place.

§ **2.** In ten days of splendid and conclusive experiment in the autumn of 1831, Faraday had therefore not only discovered the law of induction of currents, but the facts of magneto-electricity as well; and more, for he had not merely accumulated a mass of experimental results, but had reduced the whole valuable store of knowledge to one fundamental principle of exquisite simplicity, namely, that the passage of a line of magnetic force *across* a line of a conducting circuit generates in that portion of the circuit an electromotive force, or force setting electricity, or tending to set electricity, in motion.

The subsequent work of all experimentalists and mathematicians has been to work out the applications of this principle in countless forms; but no one has since added any essential discovery of fact which is not implicitly contained in the series of discoveries by which, in this short space, Faraday stepped from happy conjectures into possession of facts, which have proved more fertile in far-reaching practical consequences than any of those which even his genius bestowed upon the world. Faraday's theoretical views, however, on the phenomena underwent, in process of time, some modification. He apparently distinguished at first between the induction of currents by a current, which he called *volta-electric induction*, and the production of currents by a conductor moving in a magnetic field, which he called *magneto-electric induction*. That which seemed to impress him most forcibly was, however, the fact that it was only the beginning and ending of the inducing current which had any effect upon the other circuit. He considered that, since the mere cessation of the inducing current was accompanied by a wave of induced current, that could only be because the induced current circuit was, meantime, in a peculiar condition, to which he gave the name of the *electro-tonic state*, the annulment of which gave rise to a current in the circuit. The same state he considered to be found in a wire or circuit at rest in a magnetic field. The circuit was in the electro-tonic state whilst in the field, but withdrawing the circuit or removing the magnetic field annulled the electro-tonic state and gave rise to a current. To use his own words at a later date (Ser. XXVIII., § 3172, "Exp. Researches"), "Mere motion would not generate a relation which had not a foundation in the existence of some previous state;" and (Ser. XXIX., § 3269, *ibid.*) "Again and again the idea of an electro-tonic state has been forced upon my mind." The mere motion of an external body, such as a copper wire, in a magnetic field cannot, he considers, be the sole cause of the current, unless there is a previous peculiar state as regards the wire which, when motion is superadded, produces the current. When, however, subsequent thought and diverse experiment had clarified his ideas and adjusted facts in proper relation, he came to see that that which he had denominated the *electro-tonic state* is really the amount of electro-magnetic momentum which the circuit possesses in virtue of its being in a magnetic field. In modern language, it is the equivalent of that which is now called *the number of lines of magnetic force* passing through the circuit. Every line of magnetic force is a closed loop or continuous line, and if we set-out at any point on a line of magnetic force and travel forwards along that line we shall come back to that same point again. If this line of force is originated by a permanent magnet or an electro-magnet, then part of our journey will be performed through the iron or steel and part through the air or other diamagnetic surrounding it. If, then, a closed conducting circuit is so situated that the line of force considered passes through it or is linked with it, the line of force and the closed circuit form, as it were, two links of a chain, and cannot be separated except by pulling one through the other (*see* figure). When they are so pulled through one another, the line of force "cuts" and is cut by the circuit. The number of lines of force, therefore, which at any instant are linked with a given circuit represent potentially the greatest amount of "cutting" possible. The existence of lines of magnetic force linked with the circuit is an essential antecedent to the appearance of a current of induction

in that circuit when removed from the magnetic field. At a later stage of his investigations, Faraday was able to modify his earlier notions of the electro-tonic state, and learnt to look on the induced current appearing under these circumstances as due not to a state of things *in* the circuit, but to a condition of things *outside* the circuit, or, more precisely, to the relation in which the circuit stands to the magnetic field of force around it.

In the 28th and 29th series of his "Experimental Researches" Faraday exhausted all possible means of experiment in proving that this conception of the linking or unlinking of loops of force and loops of conducting circuits was an unerring guide to the solution of all problems of electromagnetic induction. The circuit being given, he was able to show by a course of rigid demonstration that the process of linking with it a loop of magnetic force was always accompanied by the passage of a wave of current round the circuit in one direction, and the unlinking was invariably associated with the flow of an opposite pulsation of electricity. Moreover, and most important of all, he built up a quantitative conception around the term "a line of magnetic force," so that it came to him to mean not merely a geometrical line or a direction, but a definite physical magnitude, which represented the product of a certain area of space, and a certain mean intensity of magnetic force over that area.* Armed with this idea, he proceeded to show that the quantity of electricity represented by each current of induction is the numerical equivalent of the "number of lines of force" which are linked or unlinked with the circuit by any operation. He found that this hypothesis never failed to enable him to render a satisfactory and a logical explanation of all his results, and with this clue in hand he could find his way about amidst the entanglements of experimental inquiry, and return always from each fresh excursion after fact with new confirmation of its consistency, and with fresh power to predict results of other experiments.

So strong became at last his conviction that these lines of force could hardly have such powers if they were mere geometrical conceptions, like lines of latitude and longitude, that he gives expression to it by speaking of them as *physical lines of force.* He intends to imply that he thinks "a line of force" must be taken to be a definite *something* going on in a certain region of space, and that whatever may be its real nature, we must accord to it a definite physical character in some sort or sense, as much as we do an electric current of unit strength flowing along a prescribed circuit. Faraday was not a professed mathematician, and it was perhaps fortunate that his inability to employ the mechanical aid of symbolic reasoning forced him to make clear to himself each step by experimental demonstration. He was thereby compelled to keep to the main track of discovery and prevented from deviating into the more abstract lines of thought.

* Faraday's notion of "a line of force" was at first merely a geometrical conception, representing a certain line of action, but his ultimate applications of the term showed that he had come to think of it as a *surface integral.*

The special abilities of Thomson and Helmholtz, and subsequently those of Clerk Maxwell, were, however, directed to the complete elucidation of these conceptions of Faraday, and the great treatise of Maxwell, as he himself has stated, was undertaken mainly with the hope of making these ideas the basis of a mathematical method. The one cardinal principle which may be said to be at the basis of the mode of viewing electrical and magnetic phenomena introduced by these investigators is the denial of action at finite distances, and accounting for the phenomena by the assumption of the existence of a *medium* which is the active agent in the transmission of energy from one place to another, and which is itself capable of storing up energy in a potential and kinetic form.

The mathematical methods and hypotheses of the French school of physicists, represented chiefly by Ampère, Arago, Poisson and Coulomb, consisted in the assumption that material particles in special states, called electric and magnetic, could act on one another at finite distances without any intervening mechanism according to certain laws of force varying with the distance. Faraday may be said to have raised the standard of revolt against this notion, and indeed he was able to quote in his support the great authority of Newton in rejecting the idea that matter could act on matter across intervening distance without aid from any mechanism. He never considers bodies as existing with nothing between them but their distance, and acting on one another according to some function of that distance. He conceives all space as a field of force, the lines of force being in general curved, and those due to any body extending from it on all sides, their direction being modified by the presence of other bodies. A magnet, an electrified conductor, or a wire conveying an electric current, are thus the focus and originators of a system of radiations of force lines or loops, which are to be thought of as part and parcel of it. This force system is capable of deformation or change by the presence of other bodies, but it moves with the magnet, electrified body, or current-carrying wire. These force radiations penetrate surrounding bodies, and the apparent actions between bodies at a distance are in reality actions due to immediate action of the field of force of one body upon the other at the place where it is. Then rises for solution the important problem—What *are* these lines of force? Faraday answered the question by saying that they consist in some sort of operation or action going on in a *medium* along certain lines or axes, and Maxwell added to this the suggestion that the electromagnetic medium must be identical with the medium postulated to account for the phenomena of light.

The question which yet remains unanswered is—What *is* that action or operation along certain lines in this medium which causes a line of force to exist? The future of electric and magnetic investigation will perhaps conduct us step by step to the solution of this supremely important problem.

CHAPTER II.

ELECTRO-MAGNETIC INDUCTION.

§ 1. **Magnetic Force and Magnetic Induction.**—A small magnetised steel needle freely suspended at its centre of gravity, when placed at any part of the earth's surface except just over the terrestrial magnetic poles, sets itself in a certain azimuth and position. It is found that there is a certain direction in the needle round which it can be revolved without changing the *set* of that line when the needle is left free to obey the terrestrial magnetic force. The direction of this line in the needle is called its *magnetic axis*. If this needle is placed in the neighbourhood of a magnet or a conductor conveying an electric current, it will be found that when freely suspended as before the magnetic axis takes a certain direction. This direction is called the direction of the *magnetic force* at that point in the air. If the needle be carried about in the neighbourhood of magnets or conductors conveying electric currents, this direction-imposing influence is found to affect it in various positions. The region round these active agents is called a *magnetic field*. Let this exploring needle be carried along a path moving in such a manner that each small step of its motion is in the direction of its magnetic axis, the centre of this needle will trace out a line which has the property that its tangent at any point is in the direction of the magnetic force at that point; such a line is called a *line of magnetic force*. The whole of a magnetic field may be imagined to be filled up with closely described continuous lines of magnetic force. A very thin steel wire so magnetised that equal portions of it possess identical magnetic qualities is said to be uniformly magnetised. If such a uniform magnetic filament is broken in the middle, the ends present equal and oppositely named magnetic poles. If these poles, when placed at a unit of distance apart in air, attract with a unit of force, they are said to be unit magnetic poles.

A unit north-seeking magnetic pole placed in a magnetic field, the other pole being so far removed as not to be affected, will tend to move along the lines of force; and the direction in which a free north pole tends to move in a magnetic field is called the *positive direction of the lines of magnetic force.**

The force, measured dynamically, which acts on such a free unit pole is called the *numerical value of the magnetic force* at that point, or otherwise the *strength of the magnetic field.*

If a piece of unmagnetised iron, steel or other paramagnetic metal is placed in a magnetic field it is traversed by lines of magnetic force, and it acquires magnetic polarity and exhibits magnetic qualities whilst in the field. This effect is called magnetisation by induction. If a very small portion of the paramagnetic metal could be cut out and retain the same magnetic condition it has *in situ*, it would be found to possess a certain magnetic axis. The direction of this axis is called the direction of the magnetisation of that element or small portion at that part of the mass so examined. An electro-magnet is a term applied to denote a mass of iron which is magnetised by induction, the field being created by a conductor conveying a current. Let an electro-magnet or mass of soft iron under induction be supposed to have a very narrow cut or crevasse made across it perpendicularly to the direction of its induced magnetisation at that place, and suppose a free unit pole is placed in this air gap or cut, the force which acts upon this free unit pole placed in the gap will be one compounded of the force due to the field and that due to the distribution of free magnetism which makes its appearance on the faces of the cut or air gap. This actual force on the unit pole so situated is

* If a short piece of steel could be obtained having a magnetic pole only at one end, such would be called a *free magnetic pole*. If placed in a magnetic field it would be accelerated along the lines of magnetic force just as a heavy body is accelerated along the earth's lines of gravitational force. The strength of the magnetic field is analogous to the acceleration of gravity (denoted by g), and the strength of the magnetic pole would correspond to the mass of the falling body. The impressed force is numerically equal to the product of the strengths of the pole and the field.

called *the magnetic induction* in that space. If such transverse cavities are made in the iron under induction in other places we can similarly measure the magnetic induction at those places.

The magnetic induction is a quantity which has at all points in the interior of a magnet a certain direction and magnitude, and the direction of this magnetic induction may be delineated by *lines of magnetic induction* drawn in similar fashion to the lines of magnetic force.

It should be borne in mind that in the air space outside a magnet or a mass of iron under induction, the magnetic force and magnetic induction have the same direction and mean the same thing, but inside a magnet or mass of iron under induction, they must be distinguished. The magnetic force outside the magnet may be called the magnetic induction through the air, and generally in the non-magnetic material surrounding the magnet the magnetic force and magnetic induction are one and the same. In the interior of a mass of iron under induction in a magnetic field, the magnetic force at each point is one compounded of that due to the external or original field and that due to the induced polarity acquired, and which acts to produce an opposing magnetic force. Hence the effect of the induced poles on any element in the interior of the iron is to tend to demagnetise it when the external magnetising force is withdrawn. At any point where there is no magnetism, the magnetic force and the magnetic induction are identical, but inside a magnet they are not the same in magnitude, and often not identical either in direction. In the inside of a straight uniformly magnetised bar the magnetic force due to the influence of the poles themselves is *from* the end which points to the north *to* the end which points to the south, both within the magnet and in the space outside. The magnetic induction, on the other hand, is from the north pole to the south pole *outside* the magnet, and from the south pole to the north pole *inside* the magnet. A line of induction followed round, moving always in the positive direction, is found to be a closed loop or endless line. It is a fact of fundamental importance that a thin disc of iron or steel magnetised so that at all points the magnetic axis of each small element of it is in a direction normal to its surface produces a magnetic field identical with that produced by a wire conveying an electric current coinciding in form with the edge of the disc. In short, the magnetic field of a closed circuit conveying a current is identical with that of a *magnetic shell* filling up the aperture of the circuit. The magnetic field due to electric currents circulating in conductors is, however, of such a nature that each line of induction *embraces* or surrounds the axis of the circuit once or more. The magnetic field due to permanent or electro magnets is of such a nature that each line of induction *passes through* the magnet, giving rise to magnetic polarity at the places where it enters and leaves the iron or steel. Every line of induction either surrounds an electric current or passes through magnetised iron.

The *intensity of magnetisation* of any element of a magnet or mass of iron under induction is a term requiring definition.

The couple required to hold a very small magnet when placed with its axis of magnetisation perpendicularly across the lines of force of a uniform magnetic field in air of unit strength is a numerical measure of the *moment* of the magnet.

The moment of a magnet or of any element of a magnet may be considered numerically to be made up of two factors—one, its volume, and the other its *intensity of magnetisation*, or simply its *magnetisation*; and hence, for a uniformly magnetised small linear needle, we may define the intensity of its magnetisation by saying that it is the magnetic moment of unit volume of it. Intensity of magnetisation is, like force and induction, a vector quantity.* In the case of a very long thin wire of soft iron placed along the lines of force of a uniform field the three quantities—the magnetic force, the magnetic induction, and the magnetisation—are all in alignment at any point.

It is important that the full meaning of the phrase "magnetic force at any point" should clearly be grasped. If there be any uniform magnetic field of strength H_0, and in it is placed a mass of iron in the shape of an elongated bar, the configuration of this uniform field is disturbed and magnetic polarity is developed in the iron. At any point in the interior of the iron there is a magnetising force, henceforth denoted by

* A vector quantity is one which is only precisely fixed when we know its direction as well as magnitude.

the letter H, which is due partly to the original magnetic field H_0 and partly to the induced poles which create a force opposing H_0. This resultant magnetic force is spoken of as the magnetising force in the iron, and it is the resultant of the external magnetic forces and the internal magnetic forces due to the polarity. If the form of this bar is so chosen that there are no magnetic poles, as in the case of a ring lapped over with an endless solenoid, then the magnetic force in the iron is simply calculable and it is that due to the external force alone.

At each point in the iron the *magnetic force* H must be thought of as producing *magnetisation* or magnetic displacement, just as an electrostatic phenomena electric force produces in a dielectric electric displacement or electric strain, or just as mechanical stress produces in an elastic body ordinary strain or displacement at every point. This magnetisation is not necessarily in the same direction as the force. If a narrow crevasse is made in the iron in the direction of the magnetisation at that point, the force on a unit pole held there is simply that due to the resultant force. If, on the other hand, the crevasse is made perpendicularly to the magnetisation, the force B on a unit pole held in it is one due to the joint action of the resultant magnetic force H and that due to the free magnetism or magnetic polarity developed on the walls of the crevasse, and it is now called *the magnetic induction*. The induction has at each point a certain magnitude and a certain direction, and can be represented, like the magnetic force, by lines of induction. The simplest mode of presenting the definitions of the induction B and the magnetisation I when uniform is to say that B is the "number of lines of force" (common usage) per square centimetre which would be found if we were to cut an extremely narrow crevasse perpendicular to the direction of magnetisation, and I, the intensity of magnetisation, is the magnetic moment of the metal per cubic centimetre.

A small needle-shaped fragment or flat disc of soft iron freely suspended in a magnetic field places or tends to place its greatest length along the lines of induction in that field without regard to their direction, and the well-known method of delineating what are usually called the lines of force of a magnet with iron filings is in reality rendering visible the paths of these lines of induction.

The lines of induction of a permanent steel magnet are then to be thought of as closed loops which pass in their course partly through the steel and partly through the air. The lines of induction of a circuit conveying an electric current are closed loops entirely surrounding the axis of the wire. If this circuit is a straight wire, with the return wire at a very great distance, the lines of induction are concentric circles described on a plane perpendicular to the axis of the wire and having their centre in that axis.

If a circuit is formed by coiling up into a circular coil a length of insulated wire, the coil having n turns, then to a first

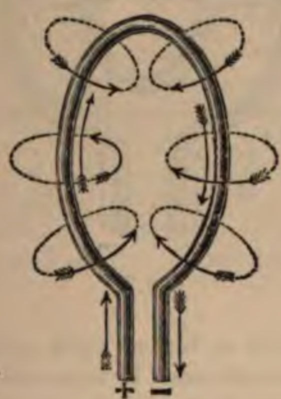

Fig. 1.

degree of approximation we may say that each line of induction forms a closed curve embracing the circuit n times. Thus, if the wire forms a coil of one turn (Fig. 1) each line of induction (represented by the dotted line) is a closed loop embracing or linked with the circuit once. If we take a circuit of two turns (Fig. 2) then nearly all loops of induction belonging to one single turn embrace not only that turn but the adjacent turn, and if the circuit could be supposed to be opened out straight, without destroying (Fig. 3) the loop of induction, it would be found to be twisted *twice* round that circuit. By similar reasoning, if a loop of induction embraces or is linked with n turns of a conducting circuit, it is in fact the same as linking each loop of induction n times with the single circuit.

Let a conducting circuit have the form of a helix (Fig. 4), then the lines of induction are closed loops, which embrace some or all the turns of the spiral, and if the helix have *n* turns then each loop of induction, according to its length and position, in reality embraces that circuit 1, 2, 3......or *n* times. If there

Fig. 2.

be two circuits, in one or both of which currents are flowing, then each circuit is surrounded by lines or loops of induction, and of those belonging to one circuit some or all are linked in as well with the other circuit, so that a certain number of all

Fig. 3.

the loops of induction are common to the two circuits, and are called the loops or lines of mutual induction.

When an electric current begins to flow in a circuit it does not, as we shall see later (Chap. III.), rise up to its full strength at once, but increases gradually. During this period of growth of the current it is necessary to regard the lines of induction which surround the wire as growing outwards, by a motion which resembles the outward expansion of the ripples on the surface of still water when a stone is dropped into it. The first formed ripples expand outwards and fresh ones make their appearance at the centre, precisely as in the phenomena called Newton's rings, in which coloured circles of light are produced by pressing a slightly convex lens against a plate of glass, it is noticed that more or less pressure makes the rings expand or contract, new ones being produced at the centre on expansion and absorbed at the centre during contraction.

Hence we must regard an increasing or a decreasing current in a circuit as producing an expansion or contraction of the

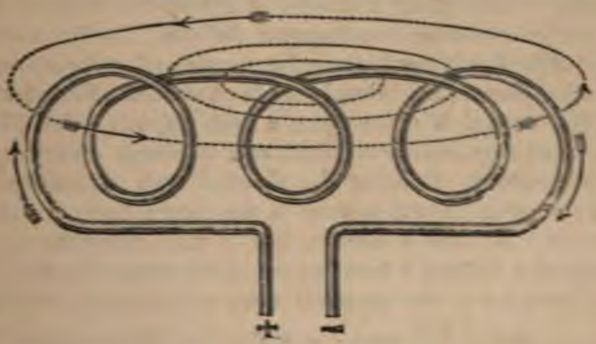

Fig. 4.

lines of magnetic induction, fresh rings or loops of induction being shed off from or taking their rise in the circuit, and the already existing ones being either expanded out or more compacted together. A pulsatory current is therefore accompanied by a pulsation of the lines of induction of the nature of an expansion or contraction. Each line or loop of induction behaves as if it were an elastic ring subjected to more or less pressure from within.

If any line be drawn in a magnetic field either in air or in the interior of a mass of iron under induction, and if this line is divided into very small elements of length and the sum taken of the length of each small element, each multiplied by the magnitude of the magnetic force at its centre, estimated in the direction of its length, this sum so formed is called the ***line integral of the magnetic force*** along that line.

If $d s$ is the element of length, and ϕ the angle which the magnetic force H at its centre makes with the direction of $d s$, then the line integral of the magnetic force along the line is

$$\int \mathsf{H} \cos \phi \, d s$$

integrated between the proper limits. It is obvious that this expression represents the *work* done in carrying a free unit magnetic pole along this path.

Two cases of importance often arise. First, when the line integral is taken along a closed line or loop in a magnetic field drawn in air or other non-magnetic medium. In this case the value of the line integral is zero, because no work is done in carrying a free pole around a closed path in an air field. Second, when the line integral is taken along a path which is a closed loop, and which surrounds or is linked with a circuit conveying an electric current. Consider the simplest case. Let a straight wire convey an electric current C, the return being at a great distance. Describe a circular line of force round the wire in the air at a distance r from the axis of the wire. The length of this line is $2 \pi r$; the magnetic force at a distance r from a straight wire is $\frac{2 \mathrm{C}}{r}$ units; and the line integral along the line of force is $2 \pi r \times \frac{2 \mathrm{C}}{r} = 4 \pi \mathrm{C}$. Hence the line integral of the magnetic force taken once round the circuit is 4π times the total current through the line of force. This can be shown to be generally true, and is the general relation between magnetic force and current.*

If a line of magnetic force threads its way through a helical current which wraps itself round the line n times, as in Fig. 4, then if A amperes traverse the conductor the total quantity of current flowing through the loop of force is n A (equal to the ampere turns), or in absolute C.-G.-S. measurement is $\frac{n \mathrm{A}}{10}$;

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* See *Electrician*, Vol. X., p. 7: Mr. Oliver Heaviside on "The Relation between Magnetic Force and Electric Current."

hence the line integral of the magnetic force taken along any loop of force threading n times through the circuit is $\frac{4\pi}{10} n A$.

The line integral of magnetic force round a closed line has been called by Mr. Bosanquet and others the *magneto-motive force* for that magnetic circuit.*

§ 2. **Tubes of Magnetic Induction.**—Faraday and Maxwell have raised the conception of a line of magnetic induction from a simply directive notion to one which enables it to be used to convey a quantitative knowledge of the magnetic field—in other words, have enabled lines of induction to be used not only to show the direction of the induction, but also its magnitude in certain units. By this means the magnetic field can be mapped out into areas and volumes which have a definite dynamical signification.

In the 28th series of his "Researches on Electricity" Faraday has gathered together his ideas on magnetic lines of force, and by a series of researches inimitable for physical insight and exquisite experimental skill has shown how they possess a quantitative as well as a directive application in all problems in which a magnetic field is considered. He lays stress in the first place on the fact stated above that every line of force (induction) is an endless loop (§ 3,117, "Exp. Res."):—"Every line of force must therefore be considered as a closed circuit passing in some part of its course through a magnet, and having an equal amount of force in every part of its course. There exist lines of force within the magnet of the same nature as those without. What is more, they are exactly equal in *amount* to those without. They have a relation in direction to those without, and are, in fact, continuations of them."

Let a magnetic field have drawn in it a number of closely contiguous lines of induction. None of these lines can cut each other, because the resultant magnetic induction at any point can have only one definite direction.

In any region it is possible to describe a surface perpendicular to all the lines of induction. Such a surface is called a level surface.

* Mr. Bosanquet on "Magneto-motive Force," *Phil. Mag.*, V. Series, Vol. XV., 1883, p. 205.

In the case of a straight infinite current, these level surfaces will be planes radiating out from the axis of the wire, and their traces on a plane perpendicular to the axis of the wire will be a series of radial lines cutting all the circular loops of induction normally. Let A (Fig. 5) represent such a level surface, and let B be another, both cutting the same sheaf of magnetic lines of induction.

On the level surface A let any unit of area a be taken, and let this area a be projected on to the adjacent level surface B by lines of induction drawn through its boundary. We have then a tubular surface, the ends of which are formed of portions of level surfaces, and the rest of the tubular surface may be conceived to be formed of lines of magnetic induction, supposed

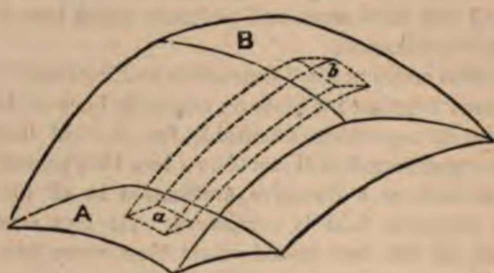

Fig. 5.

to be very closely drawn through the bounding line. Such a geometrical conception is called a *tube of induction*. The characteristic quality of a tube of induction is as follows. If the areas of the sections made by the two level surfaces A and B be called s and s', and if **B** be the mean magnetic induction over s, and **B**$'$ that over s', then **B** $s =$ **B**$'$ s', or the product of a normal cross section of tube and mean magnetic induction over that section is constant for all sections of the tube.*

If any level surface be drawn, and on this surface be marked off contiguous small areas such that the magnitude of the area is inversely as the mean value of the magnetic induc-

* Here and henceforth we shall use the thick or block capital letters to denote vector quantities. Maxwell uses the old English characters, but the block letters are easier to read and to print.

tion over that little area, and if s and B are, as before, the numerical values of any small area and the mean induction over it, then the product $\mathsf{B}\,s$ may be made equal to unity for each of these portions of that level surface. From these small areas let tubes of induction be supposed to take their rise, the whole field will be cut up into contiguous tubes of induction. Each of these tubes is called a *unit tube of induction.* By their mode of description these tubes will have small cross section at places where the field is strong and widen out in section at places where it is weak, and by the fundamental property of the tubes the value of the mean magnetic induction at any place is inversely as the cross section of the tubes of induction force at that place.

From Faraday's point of view, a magnet of any form must be mentally pictured as surrounded with and as having its whole field filled up by a closely packed arrangement of such unit tubes of induction, the tubes being intersected at right angles by the equipotential or level surfaces, and each having at any point a normal cross section which is inversely as the magnetic induction at that point. This system of tubes must be supposed to be rigidly attached to the magnet and to move with it wherever it goes. Furthermore, in accordance with Faraday's conception, each tube is an endless tube, or, as it were, a pipe returning into itself and passing in some part of its course through a magnet or round an electric current. In constructing what may seem to the student to be a highly artificial conception, we are not postulating *necessarily* any physical existence for these tubes. They should be regarded simply as a device for plotting out the space round a magnet according to a definite rule, and may, in the first place, be regarded as no more than analogous to such subdivisions of the earth's surface as we make by lines of latitude and longitude.

§ 3. **The Magnetic Induction.**—Having thus divided up a magnetic field into unit tubes of induction, it is simpler in thought to suppose a single line of induction to run down the axis of each tube and then to mentally disregard the tubular system, and, instead of speaking of a unit tube, to speak of each as a single line of induction. If we imagine a system of induction tubes starting from an equipotential surface and draw any irregular curve on this surface, we shall find that this curve encloses a certain number of tubes or lines of induction (Fig. 6). Bearing in mind that the cross section s of the tube where it sprouts out from the equipotential surface is inversely as the magnetic induction **B** at the centre of this cross section, it is at once evident that the greater the average induction over the area defined so much the more numerous will be the number of tubes or lines of induction which pass through it. If the cross section s of each tube should happen to be equal, and there be n tubes passing altogether through an area equal

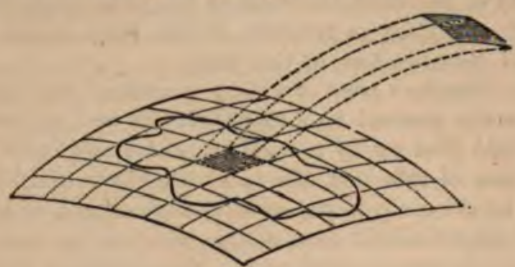

FIG. 6.

to S, bounded by the black line, then by the very definition of a unit tube

$$\mathbf{B}\, s = 1,$$
$$\text{or} \quad n\, \mathbf{B}\, s = n;$$
$$\text{but} \quad n\, s = \mathrm{S},$$
$$\text{hence} \quad \mathbf{B}\, \mathrm{S} = n;$$

and the number of tubes passing through any area S on such an equipotential surface in a uniform field is numerically equal to the product of the whole area and of the induction at any point on that area.

The product **B** S and quantities of a like nature such as is represented by the symbol Σ **B** s, indicating that the sum is to be taken of a number of elements, each consisting of the normal induction at the centre of a small area multiplied by the magnitude of that area, is called by Maxwell *the magnetic induction* through that area, and by others (Mascart and Joubert) the flux or flow of magnetic induction through that area. The following expressions all signify the same thing :—

1. The number of unit tubes of induction passing through an area.

2. The number of lines of force (induction) passing through an area (Faraday).

3. The (total) magnetic induction through an area (Maxwell).

4. The flux or flow of magnetic induction through an area (Mascart and Joubert).

5. The surface integral of magnetic induction over an area.

The surface integral of magnetic induction over an area may be more specifically defined thus. Let the induction at any point in the area be estimated, and its component taken perpendicular to the surface of that area at each point, then the product of each element of area and the normal induction over it is to be summed up over all the area, and is called the surface integral of the magnetic induction.

The characteristic quality of a tube of induction is that the flux of induction is constant throughout its length. The product $\mathsf{B}\,s$ = induction × cross section is constant, and since what is true of one tube is true of all, we may say that in a space wholly made up of tubes of magnetic induction, the total magnetic induction or flux of induction is the same across all sections of this mass of tubes. The path of a line of induction is called a *magnetic circuit*. If that path is wholly through air, for example, it is called an air magnetic circuit; if partly through iron or steel it is called an open iron circuit; if wholly in iron or steel it is called a closed magnetic iron circuit.

If a very long and thin iron wire have a length l and a cross section s and be placed along the lines of force of a uniform field of strength H, it acquires by induction a pole of strength m at each end. The poles being very far removed, the magnetic force at the centre of the wire is that due to the field alone. If the wire is cut in the middle the total number of lines of force passing across the air gap is the total induction, and it is numerically equal to the sum of the lines due to the field alone which would pass through that space if the iron was not there, and the lines added by the magnetisation of the iron. We have, then, to find the number of lines of force coming out from a pole of strength m situated at the end of a long magnetic filament.

Suppose a very small sphere of radius r described round this magnetic pole of strength m, the magnetic force at a distance r is $\frac{m}{r^2}$, and the surface integral of this force over the sphere is $4\,\pi\,r^2\frac{m}{r^2}=4\,\pi\,m$.

The total number of lines of force coming out of the pole is then $4\,\pi\,m$. Since the section of the iron wire is s, and H is the strength of the field in which it is placed, the number of lines of force passing through the median section of this wire when under induction in the field H, due to the *field alone*, is $\mathsf{H}\,s$, and the number *added* by the magnetisation of the iron by induction is $4\,\pi\,m$. Hence the total number of lines of force through the iron due to the field and to the magnetisation of the iron is $\mathsf{H}\,s+4\,\pi\,m$. This is called the total induction through the iron, and if B denote the induction per unit of cross section we have

$$\mathsf{B}\,s=\mathsf{H}\,s+4\,\pi\,m,$$

or

$$\mathsf{B}=\mathsf{H}+\frac{4\,\pi\,m\,l}{s\,l},$$

$$=\mathsf{H}+4\,\pi\,\mathsf{I},$$

where I is the average intensity of magnetisation of the iron, or is the moment $m\,l$ divided by the volume $s\,l$ of the iron.

The intensity of the induced magnetisation is a quantity of like dimensions to the strength of the inducing field H, and hence I is related to H in a definite numerical ratio. The ratio of I to H, or of the induced magnetisation to the magnetic force producing it, is called the *magnetic susceptibility* of the iron. It is denoted by κ, and we can write

$$\mathsf{I}=\kappa\,\mathsf{H}.$$

Hence, by substitution, we have

$$\mathsf{B}=\mathsf{H}+4\,\pi\,\kappa\,\mathsf{H}$$

$$=\mathsf{H}\,(1+4\,\pi\,\kappa)$$

$$=\mu\,\mathsf{H}.$$

The quantity $1+4\,\pi\,\kappa$ (denoted by μ) is the factor which expresses the ratio of the magnetic induction to the magnetising force producing it, and it is called the *magnetic permeability* of the iron.

In air and all ordinary diamagnetic media the value of μ does not differ sensibly from unity. In iron, nickel, cobalt and the group of metals called the ferro-magnetic metals the value is a considerable positive quantity.

These two fundamental equations,

$$B = H + 4\pi I$$

and

$$B = \mu H$$

are capable of being established by processes of general reasoning,* and they are to be considered as vector equations—that is to say, the first is a symbolic statement of the fact that at any point in the iron the magnetic induction is the resultant of the magnetic force, and 4π times the magnetisation.

Experiment shows that the ratio of B to H, expressed by the quantity μ, is not of a determinate character, and that the value of μ, so far from being constant, is dependent on the whole previous magnetic history of the iron, on the value of B, and on the nature of the magnetic changes the iron is undergoing, viz., whether H is increasing or diminishing.

In a cycle of operations, during which a bar of iron is exposed to increasing magnetising forces and then afterwards to decreasing ones beginning and ending with the same force, the value of B is always greater on the descending than on the ascending course. This phenomenon, which is exemplified familiarly by the retention of magnetism in a bar after withdrawal of the magnetising force, is called by Prof. Ewing hysterĕsis (from ὑστερέω, to lag behind).†

The magnetic permeability, so defined, is a quantity which is in magnetism the analogue of *specific inductive capacity* in electrostatics, and of the conducting power of a body for heat. It was spoken of by Faraday as *the conducting power of a magnetic medium for lines of force* ("Exp. Researches," Ser. XXVI.,

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* *See* Maxwell, "Electricity and Magnetism," Vol. II., second edition, § 396; also Sir W. Thomson, "Electrostatics and Magnetism," § 629.

† Ewing, *Proc. Royal Soc., Lond.*, No. 216, 1881, p. 22; also No. 228, 1883, p. 123.

§ 2,797 and § 2,846). More recently the reciprocal of μ has been called the *magnetic resistance* of the medium. The induction is greater for any given magnetic circuit for a given magnetising force, the less the total magnetic resistance.

The magnetic resistance of a circuit, composed partly of iron and partly of air, is greater in proportion as the length of the air path is increased in proportion to that of the iron. This fact is strikingly shown in experiments on the magnetic induction produced in closed rings and short bars of soft iron. Thus from some curves given by Prof. Ewing, we find that in a certain soft iron ring a magnetising force of 7 C.-G.-S. units produced an induction of 11,000 C.-G.-S. units. Whereas, in the case of an iron rod, of which the length was 50 times the diameter, the same force produced an induction of only 4,000 units. In the first case the circuit was a complete iron circuit, and the resistance small. In the second case the magnetic circuit was partly iron and partly air, and therefore the magnetic resistance was much greater.

The fact, discovered experimentally by Gilbert, that a bar of soft iron held by its centre of gravity in a uniform magnetic field settles with its length parallel to the lines of force, is not explained correctly when it is said to be merely due to the property of magnetic induction, in virtue of which the bar of soft iron becomes temporarily a magnet. For exactly the same statement would be applicable to a row of soft iron balls rigidly connected by a non-magnetic frame; yet such an arrangement would not experience any directional tendency. An elongated magnetisable mass of material tends to place its greatest length parallel to the lines of magnetic force because it becomes more intensely magnetised in that position than if placed across them. In other words, it experiences a greater magnetic induction, and we may say that the mass settles itself, if free to move, in that position which will reduce the magnetic resistance of the circuit to a mimimum.

In the above example it has been assumed, for the purpose of giving a simple illustration, that the lines of induction produced in the iron wire pass along the wire wholly within its mass and only make their exit and entrance at the ends In reality, however, in such a case there would be a considerable *lateral leakage* of lines of induction out from the iron, so that the induction falls off as we pass from the middle to the ends of the iron wire, and the permeability cannot be considered as constant along such a wire, but only considered to be so over small elements of the length. Experiments to determine μ can only be performed properly on very long bars or rings of iron placed in calculable magnetic fields by lapping over the ring with insulated wire or placing the bar in a helix, so that an electric current traversing this wire generates a field having known values at each point in the interior of the coil. Such experiments have been carried out extensively by various experimentalists, and the results embodied in curves called *permeability curves.** The form of these permeability curves is considerably affected by the temperature, and for each magnetic metal there appears to be a temperature at and beyond which it is not much more magnetically permeable than air. The permeability of nickel and cobalt varies very much with temperature. In Fig. 7 are shown the permeability curves for iron and for nickel for two very different temperatures. At about 750°C. iron, and at about 400°C. nickel, possess a permeability not sensibly greater than air.† In cobalt, permeability appears to be increased up to about 150°C. and then diminished.

§ 4. **Rate of Change of Magnetic Induction through a Circuit.**—Since every line of magnetic induction is a closed line or loop, it follows that when a conducting circuit, such as a thin wire circuit, is placed in a magnetic field it must be thought of as being linked or looped with a certain number of

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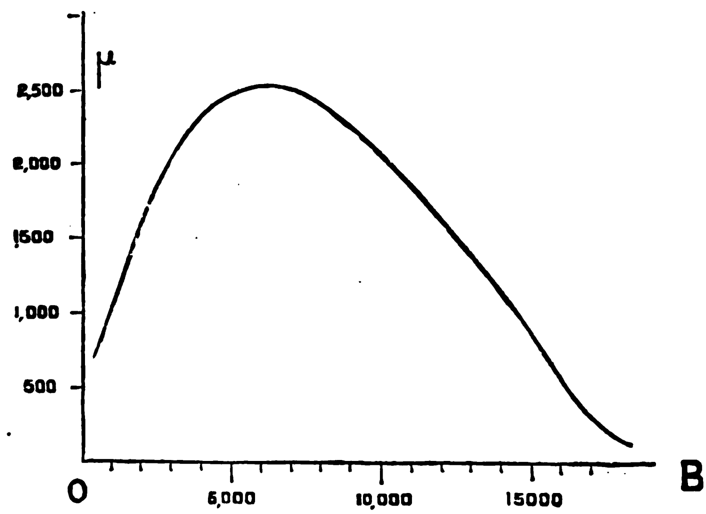
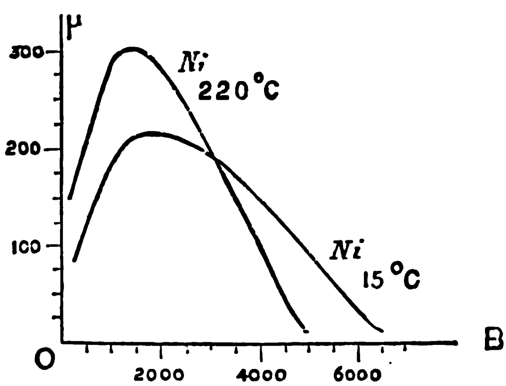
* Accounts of experiments and investigations on the form of the permeability and susceptibility curves for iron and other paramagnetic metals will be found in the following memoirs:—Weber, *Electrodynamische Maassbestimmungen*, Bd. III., § 26; Von Quintus Icilius, *Poggendorff's Annalen*, CXXI., 1864; Oberbeck, *Pogg. Ann.*, CXXXV., 1868; Riecke, *Pogg. Ann.*, CXLI., 1870; Stoletow, *Pogg. Ann.*, Ergbd. V., 1870; Rowland, *Phil. Mag.*, 1873-1874, Ser. IV., p. 336, p. 254; Bouty, *Comptes Rendus*, 1875; Fromme, *Pogg. Ann.*, Ergbd. VII., 1875; Warburg, *Wiedemann's Annalen*, XIII., 1881; Bidwell, *Proc. Roy. Soc. Lond.*, No. 245, 1886; Bidwell, *Proc. Roy. Soc. Lond.*, Vol. XLIII., 1888; Bidwell, *Proc. Roy. Soc. Lond.*, No. 242, 1886; Ewing, *Trans. Roy. Soc. Lond.*, Part II., 1885; Hopkinson, *Trans. Roy. Soc. Lond.*, Part II., 1885.

† See *Proc. Physical Soc., London*, Vol. IX., p. 187: Mr. Tomlinson on "The Temperature at which Nickel begins to Lose its Magnetic Qualities."

loops of induction. It cannot be extricated from this field without pulling the conducting circuit through lines of induc-



Permeability Curve for Iron (Rowland).



Permeability Curves for Nickel (Rowland).

FIG. 7.

tion. If the conducting circuit be linked with any definite number of lines of induction, then any movement of the conductor in any way will either cause or not cause a change in the total number of lines linked with it. It is easy to devise all sorts of movements of a conducting circuit in a field such that the motion causes no integral change in the total magnetic induction through the circuit.

In this last case, although lines of induction are taken out or unlinked from the circuit at one place, they are being inserted at another to an equal extent. If at any instant there be N lines of induction or unit tubes linked with the conducting circuit, and at a very short interval of time $d\,t$ afterwards there be a small increase to $N + d\,N$, then the change of induction is represented by the number $d\,N$ added or subtracted, and the rate of change is represented by $\frac{d\,N}{d\,t}$.

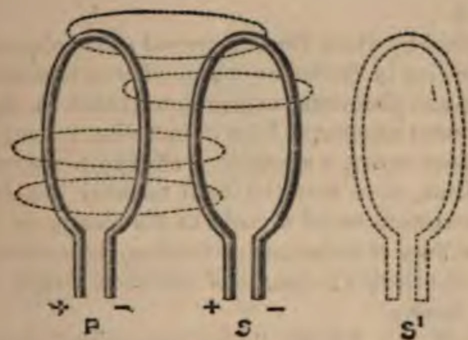


FIG. 8.

A change in the flux of induction through a circuit may be brought about in several ways. Let there be two circuits P and S (Fig. 8), and suppose that P is traversed by a current creating round it a field of induction, and having certain lines of induction linked both with itself and with the circuit S. The circuit P is linked with a certain number of lines of induction which encircle the axis of the wire or conductor. Of this number a certain proportion, or perhaps all, pass through and are linked with the circuit S. If S is moved farther away or brought nearer to P it is easy to see that if the motion is a suitable one the number of lines of induction linked with S may be diminished or increased.

Next, if the position of S is kept the same and the current in P is varied, the loops of induction surrounding P will expand out and others will make their appearance, or else will contract or shrink in and diminish in number. The number linked with S will therefore be accordingly varied.

Lastly, current remaining the same in P, and distance of S also constant, the form of S may be varied by squeezing it more or less out of shape, and thus forcing lines of induction out of it or including more in it.

The fundamental discovery of Faraday may be summed up in the following words:—

If the magnetic induction through any circuit be varied by any means, an electromotive force is set up in that circuit proportional at any instant to the rate of change of the magnetic induction at that instant.

The method by which Faraday arrived at the above fact consisted in moving in various ways loops of wire in uniform magnetic fields, and discovering experimentally that in any case in which the total number of lines of induction passing through the circuit was varied, a quantity of electricity was set flowing in the circuit, such that the total quantity put in motion by the movement varied directly as the change in the total number of lines of induction perforating the circuit, and inversely as the electric resistance of the whole circuit in which it was set flowing.

In § 3,152 and § 3,199, "Exp. Researches," Faraday has shown that the total quantity of electricity (measured by half the sine of the angle of deflection of a ballistic galvanometer inserted in the circuit) set in motion when closed loops of wire of various metals are moved in a uniform magnetic field is proportional to the conducting power of the circuit and to the total change of magnetic induction. Hence, if N stand for the number of lines of induction passing through a circuit in any position when placed in such uniform field, any small movement lasting for a time $d\,t$ which results in making a small change equal to $d\,\mathrm{N}$ in the flux of induction will start in motion a quantity, $d\,q$, of electricity, and in proper units if R be the resistance of the circuit $\frac{d\,\mathrm{N}}{\mathrm{R}} = d\,q$; but if E is the average electromotive force during this movement which is brought into existence in the circuit, and C is the strength of the average current during the interval of time $d t$, then $C\,d t = d q$ and if this circuit possesses a very small inductance, the current, as explained in the next section, will at any instant be proportional to the electromotive force, E, urging it. Hence,

$$C = \frac{E}{R},$$

$$\therefore\ \frac{E}{R}\,d t = d q = \frac{d N}{R},$$

$$\therefore\ \frac{d N}{d t} = E;$$

that is, the electromotive force during that small interval is proportional to the rate of variation of the induction.

FIG. 9.

It is necessary to arrange some conventions to connect the direction of this electromotive force with the direction of the induction. If any plane circuit is looked at from one side, the positive direction *round* that circuit is in the clockhandwise direction, and the positive direction *through* that circuit is away from the spectator.

The law of induction is as follows: If lines of induction perforate a circuit positively, *i.e.*, from the positive side, diminution of induction makes positively directed electromotive force round the circuit. Hence negative rate of change of induction creates positively directed induced E.M.F., or $-\frac{d N}{d t} = E$; that is, the differential coefficient must have the negative sign.

If during any period of time a circuit is exposed to magnetic induction the rate of change of which varies then from instant to instant the impressed inductive electromotive force varies and may be represented graphically as follows: Let the straight line O X (Fig. 10) be an axis on which lengths are marked off to represent intervals of time, and let ordinates perpendicular to it represent the instantaneous value of the flux of induction through, or lines of induction included by, a circuit, then if the variation of induction is continuous it may be represented by a curve drawn between these axes. This curve may be called a curve of induction. If at any point P a tangent P T is drawn to this curve, the trigonometrical tangent of the angle P T M represents the rate of variation of P M with respect to O M, and if P M represents at any instant the induction N, then the

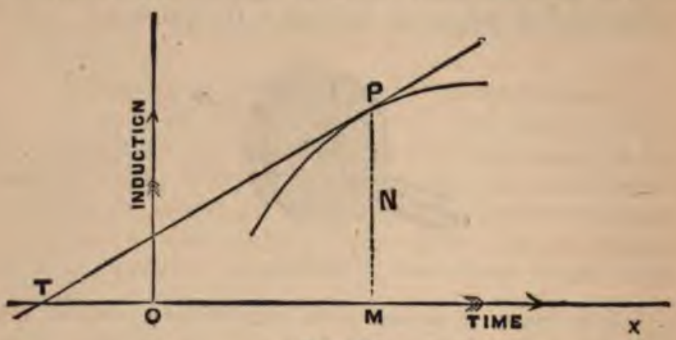


FIG. 10.

slope of the tangent at P represents $\frac{dN}{dt}$ or the rate of change of N with respect to time.

In the practical application of the above rule it must be borne in mind that if N or the number of lines or tubes is measured in units based on the centimetre, gramme, and second system, then the electromotive force is given in the same units. Since one volt is 10^8 C.-G.-S. units, to get the E.M.F. in volts we must divide the time rate of change of induction by 10^8. Thus if the change of induction be such that N C.-G.-S. lines are removed uniformly from the circuit per *minute*, the electromotive force in volts set up will be

$$\frac{N}{60} \cdot \frac{1}{10^8} \text{ volts.}$$

In practical dynamo work Mr. G. Kapp has proposed to use a unit of induction equal to 6,000 C.-G.-S. lines; hence if N′ lines of induction (in Kapp measurement) be removed or added uniformly per minute to a circuit the E.M.F. in volts will be V,

and $$V = N' . 10^{-6}$$

Mr. Kapp gives as the usual average number of (Kapp) lines of induction which pass through one square inch of cross section of iron when magnetised to approximate saturation, the following numbers* :—

Dynamo armatures—
Charcoal iron wire well annealed .... 25 lines.
Dynamo armatures—
Charcoal iron discs well annealed .... 22 „
Dynamo field magnets—
Hammered scrap .... 18 „

§ 5. **Inductance.**—In the following pages the phenomena which will chiefly concern us will be those which manifest themselves when a conducting circuit is submitted to the action of a magnetic field which varies in strength or which is traversed by an electric current which is varying or changing in magnitude. In many of the chief phenomena of electro-kinetics the observed results are dependent on a stationary electric condition having been obtained. There are, however, a large class of observed facts which are related not only to the *magnitude* of electric currents concerned in producing them, but also to the *rate of change of the magnitude* of such currents.

As long as the observed actions are those due to electric currents of constant strength, the only three electrical magnitudes necessary to consider are the strengths of the currents, the magnitudes of the impressed electromotive forces, and the electric resistances.

If the electric flow be taking place along a thin cylinder or wire parallel to the axis, and be uniformly distributed over the cross section, then it is an experimental result that the electric

* "The Predetermination of the Characteristics of Dynamos." By Gisbert Kapp. Paper read before the Society of Telegraph Engineers and Electricians. See *The Electrician*, November 12, 1886.

potential difference between any two normal sections of the cylinder is exactly proportional to the strength of the current flow within it, provided that the temperature of the conductor is constant.

The physical quality of the conductor which determines the ratio of the numerical values, according to certain chosen units, of the electric current strength and the potential differences between the chosen sections is called the electrical resistance of that volume of the conductor bounded by these sections, and the number expressing this ratio is called the numerical value of that electrical resistance

All steady current problems are vastly simplified by the fact that if this electrical resistance is determined for several values of the current strength it is found to have numerically the same value, always assuming temperature and other physical conditions to be constant.* The electrical resistance is, however, affected by change of temperature, by strain, and by various physical changes, and in some bodies, such as selenium, by mere exposure of the conductor to light. The true electrical or *Ohmic resistance* of a conductor in the linear form may be defined as follows.

The specific electrical resistance of a conductor is a quality of it, in virtue of which there is a fixed numerical ratio between the potential differences of two opposed faces of a cubic unit of it and the quantity of electricity which traverses either face per second, assuming a steady flow to take place normally to these faces, and to be uniformly distributed over them, such flow taking place solely by electromotive forces outside the volume considered.

As soon as we cease to limit our consideration to constant or steady electric currents we find that we shall not be able to give a full account of the phenomena unless we extend our ideas, and recognise another quality of conductors equally important with resistance in determining the numerical ratio of instantaneous current strength to instantaneous potential differ-

* *See* British Association Report, Glasgow, 1876. Prof. Chrystal, working under Prof. Maxwell's direction, found, as a result of prolonged experiments, that after allowing for temperature the resistance of a circuit of one ohm is not different for infinitely small currents and currents of one ampere by as much as 10^{-12} part.

ence between two points on any linear conductor traversed by that current. This quality of the circuit is called its *Inductance.*

The clear recognition of this special quality of a conductor dates from the publication of Faraday's memoir forming the Ninth Series of his "Electrical Researches" (§ 1,048, 1st Ed.), *On the Influence by Induction of an Electric Current on Itself,* and from the investigations of Prof. Joseph Henry (*Phil. Mag.,* 1840), of Princetown. The chain of experiments which led to these ideas was apparently started by the inquiry addressed to Faraday by a Mr. Jenkin one Friday evening, at the Royal Institution, as to the reason why a shock was experienced when a circuit containing an electromagnet was broken, the observer retaining in his two hands the ends of the circuit, but no shock was felt if the circuit contained neither magnet nor long coils of wire. Faraday seems speedily to have arranged an organised attack on the subject and to have returned from his investigation burdened with the spoils of victory in the shape of the following facts:—

1. If a battery circuit is closed by a short thick wire, then, although there may be a very strong current existing in this wire, on breaking contact at any point little or no spark is seen, and if the two ends of the circuit are grasped in the two hands and the interruption takes place between the hands, then little or no shock is experienced.

2. If a very long wire is used instead, then, although the absolute strength of the current may be less, yet the spark and shock at interruption are more manifest.

3. If this length of insulated wire is coiled up into a helix on a pasteboard tube, then, although the length of wire and strength of current are the same, yet the spark and shock are still more marked.

4. If the above helix has an iron core placed in it, both these effects are yet more exalted.

5. If the same length of wire is doubled upon itself, being, however, insulated, then the effects nearly vanish, and, whether straight or coiled, this doubled wire with current going up one side and down the other is no better in respect of spark and shock on interruption than a very short wire.

The first observation which Faraday makes upon the above results is that electricity would seem to circulate with something like *momentum* or inertia in the wire, and that the greater the ampere-feet (in modern language) of the current, so much the more mass of current is there to run on and jump over the obstacle presented by the first thin layer of air which is introduced between the contacts as they are separated, giving rise to a spark. He saw, however, at once, since the form of this circuit is an important factor, that the idea of inertia *in the current itself* was fallacious, or else the mere doubling the wire could not nullify all the effects. He did not at that time see that the idea of momentum was exceedingly appropriate, but its allocation in the electric current itself was wrong.

The observation which, however, led him to a consistent theory was as follows. A bobbin was prepared, having wound on it two insulated wires, 1 and 2. The ends of 2 being left unconnected, the wire 1 was used to complete a circuit, and gave a spark on interrupting a current traversing it. As we have seen (*Introductory*, Chap. I.), Faraday* had three years previously established the fact that the commencement and cessation of a current in circuit 1 would produce in circuit 2, if closed, an inverse or a direct induced electric wave or transitory current. Now, on closing circuit 2 through a galvanometer or loose contact, and interrupting a steady current flowing in circuit 1, he found that when circuit 2 was completed so that an induced or secondary current could be generated in it, little or no spark happened at the place of interruption in 1; but, if circuit 2 was opened, then the interruption of circuit 1 gave rise to a bright spark at the contact. Faraday therefore inferred that when circuit 2 was closed adjacent to circuit 1, that the current in 1 exerted its full inductive effect in generating secondary currents in 2; but that if circuit 2 was open, then, there being no adjacent conductors, the current in 1 expended its inductive effect in producing induced currents in its own circuit, and this *self-induction* manifested itself by temporarily diminishing the strength of the current at starting and assisting or increasing it momentarily at the interruption. He was thus able, from this point of view, to picture to himself the circuit of 1 as occupied by a steady current, superimposed on which was another current he called the

* Faraday's "Exp. Res.," § 1,090.

inverse extra current, lasting but a very short time at starting the steady current, and a *direct extra current* which flowed on and produced the effects of the spark or shock at the interruption of the circuit. These extra currents or currents of self-induction he found could be removed from the circuit itself and exhibited in a neighbouring circuit when that adjacent circuit was closed, and so fitted to be the seat of induced currents due to the *mutual induction* of the primary on this secondary circuit. Faraday then placed this theory under test by requiring it to furnish an explanation of the following experiment:—

M and N (Fig. 11) were two mercury cups* which formed the terminals of three circuits—a battery circuit, B, a

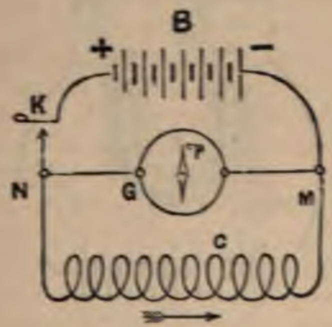


FIG. 11.

galvanometer circuit, G, and a circuit consisting of an electro-magnet or helix, C. The needle of the galvanometer was blocked in such a way that the tendency to deflect under the steady current was prevented and the needle kept at zero; but it was free to deflect in the opposite direction under an oppositely directed current. This being the case, the raising of the battery wires out of the mercury cups was accompanied by a violent "kick" or deflection of the needle in the free direction. The action could clearly be explained by supposing that after the electromotive force of the battery is removed from the coil C the current in it does not at

* Faraday's "Exp. Res.," Vol. I., § 1,079.

once stop dead, but runs on like a heavy body and makes a backwash of current through the galvanometer in the direction from M to N. An illustration of the electro-magnetic inertia of a coil on interrupting the current may be shown in a more modern form, thus. Let E (Fig. 12) be an electro-magnet, and let L be an incandescence lamp of which the resistance is very large compared with that of E. Let S be a few cells of a storage battery supplying current, and let K be a key. On depressing the key the current flows both in the magnet and in the lamp

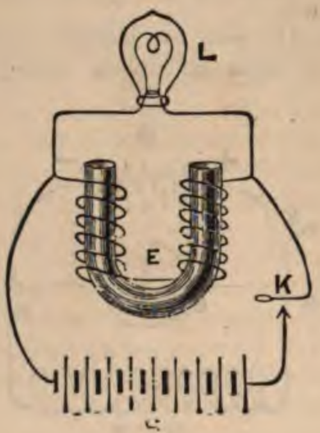


FIG. 12.

arranged as a shunt on the magnet. This current, however, is by assumption not strong enough to illuminate the lamp. On raising the key and stopping the steady current through the lamp the electric inertia of the coil sends a momentary powerful current through the lamp, which causes it to flash up. Again, if a small shunt-wound dynamo be occupied in supplying current to a few incandescence lamps, and the two hands be employed to raise simultaneously the brushes from the armature, the momentary rush of current from the field-magnet due to this *extra current* will disagreeably impress the phenomenon upon the mind of the observer if the experiment is made with any but the smallest-sized dynamo.

Neither of these experiments are well-fitted to illustrate the extra current at the closing of the circuit or the effect of electric inertia on starting the current in a helix. The arrangement most suited to exhibit the whole effect is that of the differential galvanometer as used by Edlund, or that employing Wheatstone's bridge, due to Maxwell.

In Edlund's arrangement* a differential galvanometer is employed, of which the two coils G_1 G_2 are so placed and wound that when equal and oppositely directed currents are sent through them the needle is unaffected. The coils are then

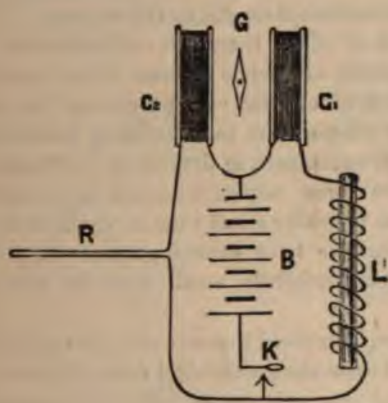


Fig. 13.

connected, as shown in Fig. 13, to a battery, B, an electro-magnet or helical coil, L, and a wire, R, of equal resistance to L, but wound double. The galvanometer coils are so connected to the circuits L and R that when the steady current from the battery flows through the divided circuit the needle remains at zero. On closing the circuit it is then found that the needle makes a sudden deflection in a direction indicating a brief current passing in coil G_2, and on breaking the circuit it makes another deflection indicating a transitory current passing through G_1. In other words, the balance is destroyed at the instant of breaking and making, but restores itself again

* *See* Poggendorff's *Annalen*, 1849.

when the currents become steady. This experiment, therefore, most clearly shows that the electro-magnetic helix L, although of exactly the same ohmic resistance as the coil R, differs from it in possessing a peculiar quality which it has in virtue of being in the form of a coil or helix, and to which the name self-induction or *inductance* has been given. We are able to define this term as follows :—The self-induction or inductance of a circuit is, speaking generally, a quality of it in virtue of which a finite and steady electromotive force applied to it cannot at once generate in it the full current due to its resistance, and when the electromotive force is withdrawn time is required for the current strength to fall to zero.

From one point of view, therefore, self-induction may be thought of as a quality of electric circuits which bears the same relation to variation of electric current strength in them that *mass* or *moment of inertia* bears to variation of linear or angular velocity in material substances in dynamical problems.

Just as a finite force cannot generate in a heavy body finite velocity in an infinitely small time in virtue of its inertia, so a finite electromotive force cannot generate in a circuit a finite current in an infinitely small time in virtue of its inductance.

It must, however, be noticed that not only does the inductance of a circuit depend upon the geometrical form of the circuit, but it depends upon the magnetic permeability of the region which surrounds the circuit and on the magnetic permeability of the conducting circuit itself. If in the arrangement with the differential galvanometer the steady balance is obtained using a copper wire helix wound on a cardboard tube and balanced against a non-inductive but equal resistance, it is found that the insertion of a soft iron core into the helix greatly increases the "kick" on making contact, indicating the passage of a greater quantity of electricity through the opposite galvanometer coil, and therefore a greater delay in the time of establishing the steady balance.

Maxwell's method of exhibiting the effect of inductance is a preferable arrangement.

Four conductors are arranged in a rectangle joining the points *a*, *b*, *c*, *d*, and the diagonals completed by a galvanometer and battery (Fig. 14). P, Q and R are non-inductive resistances, and E is an electro-magnetic helix. If R and E are equal in actual resistance and P : Q = R : E, then the permanent closing of the battery circuit does not finally affect the galvanometer indication, and these circuits (battery and galvanometer) are then said to be *conjugate circuits.*

When, however, the battery key is first put down the galvanometer receives an impulse in one direction. When the key is kept down the galvanometer soon returns to zero, or to its original position. On raising the key the needle receives an impulse in the opposite direction. Examination of these impulses shows that if the current enters the quadrangle at *d*, on closing the key the potential rises at *b* faster than it does at

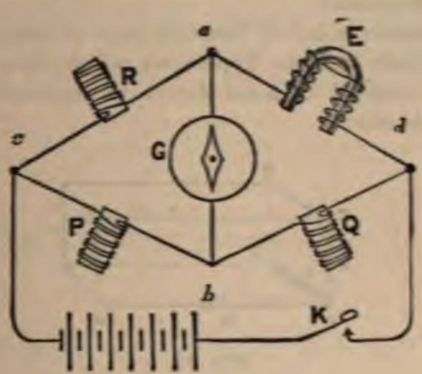


FIG. 14.

a, and that on raising the key the potential dies down at *b* faster than at *a*, but that if the "balance" is properly obtained, the points *a* and *b* reach finally the same potential when the key is kept closed.

An electro-magnetic helix, with or without a core of soft iron, behaves itself, therefore, towards an external electromotive force to which it is submitted as if it had an internal counter-electromotive force which gradually disappears—allowing the full current due to its resistance to be established in it, and behaves also at the removal of this external electromotive force as if a direct internal electromotive force suddenly made its appearance within it; this also gradually dying away.

§ 6. **The Electromotive Force of Induction.**—We have in § 4 enunciated Faraday's law of induction in terms of the variation of a quantity called the flux of induction through the circuit. It is possible to express the fundamental principle in a more elementary manner, and in a way which adapts it to explain every fact yet observed. It is as follows:—If any element of a conducting circuit is so placed in a field of magnetic induction that a movement of that element of the conductor or change in the field of induction causes lines of induction to intersect it, it creates an electromotive force in the element of which the direction is perpendicular to the plane containing the lines of magnetic force and the direction of motion of the centre of the element.

This operation is called "cutting lines of magnetic force." We shall allude later to hypotheses which have been constructed to suggest in some degree an explanation of the nature of this effect.

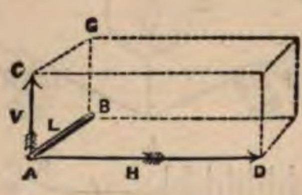


FIG. 15.

The simplest possible case which can be considered is when a short linear element, such as a straight wire, is made to move in a uniform magnetic field, in a direction perpendicular to the plane containing the field lines and the linear conductor, the direction of the length of this last being also perpendicular to the direction of the field lines.

If A B (Fig. 15) is the element of length of the conductor of length L, and if A D represent in magnitude and direction one of the lines of induction of the uniform magnetic field H, in which it is placed, A B being at right angles to A D, and if A C represent in magnitude and direction a displacement of A B taking place uniformly in one second, so that A B moves uniformly parallel to itself from position A B to position C G in one second, we have then three lines, A B, A C, A D, mutually at right angles and representing respectively the length L, the velocity V, and the magnetic induction H.

The result of the motion is to generate in A B an electromotive force E, numerically equal to the product H L V in consistent units. But since the sides of the parallelopipidon or solid rectangle are taken to represent respectively H, L and V, their product E represents the magnitude of the electromotive force of induction.

If the directions of A B, A C and A D are not orthogonal—but inclined—the same still holds good.

For let the field lines A D (Fig. 16) be supposed to be inclined at an angle θ with the direction of the length of the conductor A B, and let the direction of motion of A B parallel to itself, represented by A C, be inclined at an angle ϕ with A B.

The strength of field estimated perpendicular to A B is H sin θ, and if A C represents the actual velocity of A B or

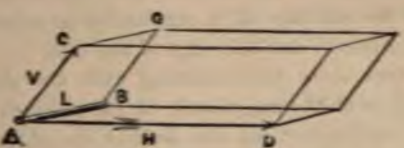


FIG. 16.

displacement in one second, then A C sin ϕ, or V sin ϕ, is its velocity in a direction perpendicular to its own length. The magnitude of the induced electromotive force E in A B is then numerically equal to L H sin θ V sin ϕ, or to H L V sin θ sin ϕ; but this expression also represents the volume of a doubly skew parallelopipidon or solid rhomboid; hence, as before, if vectors be drawn representing respectively the length and velocity of a conducting element, and also the field strength in which it is placed, the volume of the solid rhomboid described on these vectors as adjacent sides represents the magnitude of the electromotive force induced in the element.

The magnitude of this induced electromotive force is not in any way dependent upon the nature of the material of which this conductor is made. Faraday experimentally proved this ("Exp. Res.," § 193-201) by taking a double conductor composed of an iron and a copper wire twisted together and united at one end. On passing this double conductor through a magnetic field no induced current was detected in it by a galvanometer. This proved that the electromotive forces set up in each separate conductor were equal and opposite, and hence since the lengths, field, and velocities were the same no factor entered into the production of the effect, which depended on the nature of the conductor. From further experiments with circuits partly metallic and partly electrolytic fluids he inferred that in all bodies, whether what are commonly called conductors or non-conductors, or in electrolytic conductors, identically the same electromotive force is brought into existence by moving the same lengths in the same way in the same magnetic fields.

When a disc, whether metallic, or what is commonly called non-conducting, is rotated in a uniform magnetic field so that its axis of rotation is parallel to the direction of the field, there is set up a difference of potential between the centre and the edge. In the case of the metallic disc, the internal resistance being small we can tap off a current by an external wire connected to the centre and the edge of the disc.

We can now show that, starting with the elementary law above stated, as to the magnitude of the induced E.M.F. in an element of a conductor, we can deduce the other principle of the relation of the induced E.M.F. to the rate of change of the induction through the circuit.

Let A B C D (Fig. 17) be a conducting rectangle, of which the plane is perpendicular to the induction lines of a uniform magnetic field of strength H, the same being shown in plan below; let the circuit be capable of revolving about an axis O O in its own plane, and let it be displaced through any angle θ, as shown in elevation and plan in Fig. 17. If the frame is so displaced it is clear that the sides A C, B D, "cut" across lines of magnetic induction, but that the upper and lower sides do not. During this displacement the vertical sides alone will be the seat of electromotive forces. Imagine this frame to revolve round the vertical axis with a uniform angular velocity ω, and at any instant t to have a position such that its plane makes an angle θ with the plane normal to the lines of force. Let the length of the side A C be L and that of A B be R, the actual velocity of the side A C is $\frac{\omega R}{2}$, and the strength of the field in a direction perpendicular to its length and its direction of motion at that instant is H sin θ. Hence the electromotive force of induction in the side A C is $\frac{\omega R}{2}$ L H sin θ, and an equal and oppositely directed electromotive force acts in the side B D at the same instant. Hence the total electromotive force acting round the frame is equal to ω H R L sin θ. If the area of A B C D is denoted by A we may write the above as

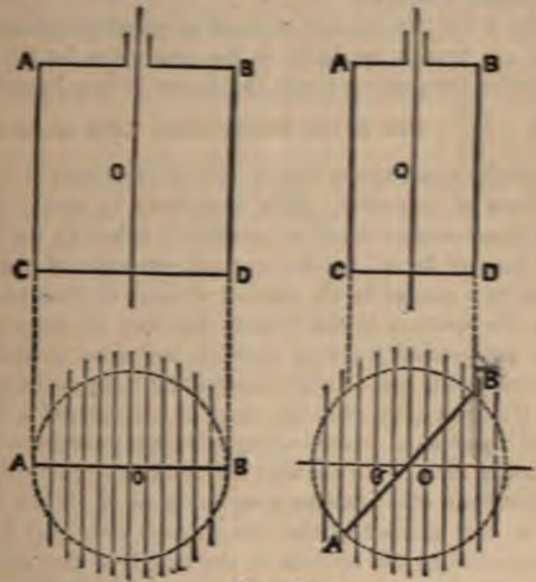


FIG. 17.

ω H A sin θ. The angular velocity ω may be expressed as the time rate of change of θ or as $\frac{d\theta}{dt}$, hence the expression for the total electromotive force of induction round the frame is H A sin $\theta \frac{d\theta}{dt}$ or $-\frac{d}{dt}$ (H A cos θ).*

The expression A cos θ denotes the apparent size of the frame as looked at from a considerable distance along the

* We here suppose the circuit to be a simple loop of wire having a negligible inductance. The above statements would require some modification for a circuit of many turns of wire.

direction of the lines of induction, and the quantity $H A \cos \theta$ is the numerical value of the number of lines of magnetic induction passing through or traversing the frame in its position when its plane is inclined at an angle θ to the normal position. We assume that these lines are spaced out according to the rule proper for such distribution, viz., that the number passing through a unit of area whose plane is taken normal to the direction of these lines is numerically equal to the magnetic induction over that area.

Writing N for this number of lines so piercing through the frame at any instant, we have, as the expression for the total electromotive force acting round the frame at any instant, the quantity $-\frac{dN}{dt}$; that is, the electromotive force of induction is numerically equal to the rate of change (decrease) of the included lines of induction. It is customary to speak of this induced electromotive force as generated either by the "cutting of lines of force" by the various elements of the conductor or by a change in the number of lines of force piercing through the aperture of the circuit; but they are merely two different geometrical ways of viewing the same phenomena. The actual results are capable of receiving a physical explanation on the assumption that the act of intersection of a line of force and a portion of a conducting circuit is productive of an electromotive force. We see that the total electromotive force is the resultant effect due to a summing-up of all the forces acting on each element of the circuit, each elemental E.M.F. being measured by the product of the length of that element, the field strength around it, and its normal velocity in that part of the field. The result is concisely expressed by the number which expresses the time rate of change of the whole number of the lines of induction traversing the circuit. This same may be extended to any circuit of any form moving in any way in any field.

If a circuit of any form which is traversed by an electric current is placed in a magnetic field due to other neighbouring currents or magnets, there is a flux of induction through that circuit due partly to the current in the conductor and partly to the external field of the other currents or magnets. If there be M lines of induction due to the external field passing through it, and N lines of induction due to its own current, any variation of the external induction, of which the rate of change at any instant is represented by $\frac{dM}{dt}$, will produce an impressed electromotive force in such a direction that taking lines of induction out of the circuit induces an electromotive force in the clockhandwise (+) direction, as seen from that side of the circuit at which the lines enter. When a current is flowing in any conductor the relation between the direction of the current and that of its own lines of induction is the same as the relation between the thrust and the twist of a corkscrew. Hence it is evident that if we consider a circular current (Fig. 18)

FIG. 13.

with the current flowing in it clockhandwise (+), as seen from one side, that its own lines of induction pass through the circuit in the positive direction, or away from the eye.

Accordingly, a little reflection shows that if the current in the conducting circuit is made to increase, an opposing electromotive force is created by the increasing induction of the current on its own circuit. The current in the act of increasing crowds its own circuit more full of lines of induction, and creates an electromotive force of induction during the period of this increase equal at any instant numerically to its own rate of increase, and directed in opposition to the direction of the impressed external electromotive force which is driving the current.

§ 7. **Electro-magnetic Momentum.**—When a heavy body is in motion it possesses at any instant *momentum*, in virtue of its inertia. Numerically the momentum of a heavy particle is obtained by taking the product of its mass and its velocity, each measured in appropriate units. The time rate of change of a body's momentum in any direction is by the second law of motion the measure of the force acting upon it in that direction, or, in the notation of the calculus,

$$\frac{d\,(m\,v)}{d\,t}=f.$$

We have seen that the induced electromotive force in a circuit depends on the time rate of change of the magnetic induction through it, and hence the magnetic induction at any instant through a circuit bears the same relation to the induced electromotive force in it that a body's momentum does to the mechanical force acting on it. Maxwell has accordingly employed the term *electro-magnetic momentum* to represent the flux of magnetic induction or the number of lines of magnetic induction passing through a circuit, because it is upon the rate of change of this quantity that the induced electromotive force depends. Faraday very early recognised that induction effects depend on a *change* of some quantity. He gave to this quantity the name of the *electrotonic state*, and he spoke of a conductor in a magnetic field, when traversed by lines of induction, as in the electrotonic state, and he considered that when the electrotonic state was either assumed or disappeared its commencement and end was marked by the production of the induced electromotive force. Consider then the operations which go on when a conducting circuit—suppose a simple loop of wire—is subjected to a steady electromotive force. The instant that force is applied a current begins to flow in the circuit; the instant that current begins lines or rings of induction spread out from the circuit; and the loop at any instant encloses a certain number of lines of induction which are increasing at that instant at a certain rate. A counter- or opposing electromotive hence exists in that circuit numerically equal to the time rate of increase of this induction. In circuits which do not enclose or surround iron or other magnetic metal, or which are immersed wholly in a medium of constant permeability, the induction at any point in the neighbourhood of the circuit is numerically proportional to the strength of the current at that instant flowing in the circuit. This is the fact which lies at the root of the operation of most galvanometers, viz., that the field at any point in the neighbourhood of the coil is simply proportional to the strength of the current flowing in the coil. If, then, i represent the strength of the current at any instant in the circuit, and L be a certain constant quantity, such that Li represents the induction through the coil or circuit due to the current i in it, then Li is the measure of the electro-magnetic momentum of that circuit. This quantity L is a coefficient which, in this case, is dependent only upon the geometrical form of the circuit, and under the assumption that there is no magnetic material in or near the circuit through which the lines of induction can pass, it is a constant quantity.

This quantity L is called the constant *coefficient of self-induction* of the circuit, or, more shortly, *the inductance* of the circuit.

We may define it thus:—

The Coefficient of Self-Induction.—In the case of circuits conveying electric currents, which are wholly made of non-magnetic material and wholly immersed in a medium of constant magnetic permeability, the total induction through the circuit per unit of current flowing in that circuit when removed from the neighbourhood of all other magnets and circuits is called the coefficient of self-induction. Otherwise the ratio of the numerical values of the electro-magnetic momentum of such circuit and the current flowing in it when totally removed from all other currents and magnets is the numerical value of the *inductance* of that circuit.

§ 8. **Electro-magnetic Energy.**—Let us confine our attention first to one circuit of constant inductance or self-induction in which a current is being generated by a constant electromotive force applied to it. Each increment of strength of the current creates an electromotive force opposing the impressed or external electromotive force. Hence this external electromotive force has to do work against an opposing force of its own creating all the time the current is rising in strength. When a mechanical force overcomes a resistance through a certain distance, mechanical work is being done, and, accordingly, we may ask—what is the electromotive force doing all the while it is increasing a current against an opposing electromotive force? The answer is, it is doing electrical work. The result of causing a current having a strength i at any instant to flow for a small time, dt, against an opposing E.M.F. at any instant equal to e, is that a quantity of work represented by $e\,i\,dt$ is done in the time dt. If e is the instantaneous opposing electromotive force of self-induction it is measured at any instant by the rate of change of electro-magnetic momentum $\mathrm{L}\,i$, or by $\mathrm{L}\frac{di}{dt}$.

Hence the work done in raising the current from a strength i to a strength $i+di$ against the counter-electromotive force of self-induction is $\mathrm{L}\frac{di}{dt}i\,dt = \mathrm{L}\,i\,di$, and if this is integrated between limits zero and I we get the whole quantity of work so done against self-induction alone in bringing up a current from zero to its full value I in the conductor, but

$$\int_0^{\mathrm{I}} \mathrm{L}\,i\,di = \tfrac{1}{2}\,\mathrm{L}\,\mathrm{I}^2.$$

The total work done against the electromotive force of self-induction in creating a current I in a conductor of constant inductance L is then numerically equal to half the square of the final current strength multiplied by the value of the constant inductance or coefficient of self-induction.

The equivalent of this work is found in the magnetic field formed round the conductor, and hence the formation of a magnetic field represents so much energy, measurable in foot-pounds per cubic inch, or in any other similar units such as ergs or kilogrammetres per cubic centimetre of field.

Next let us consider the case of two circuits. Let the constant coefficient of self-induction of the first be L, and let it be traversed at any instant by a current i. Let the inductance of the other be N, and let it be traversed by a current i'. Let the *coefficient of mutual inductance* be M.

The definition of this last quantity is as follows: If both circuits be traversed by unit currents, and if there be no other field than that due to these currents the number of lines of induction which traverse *both* circuits or are linked with both circuits is called the constant coefficient of mutual inductance. It will be a quantity constant for a given form and position of the two circuits on the assumption that the lines of induction flow in a medium of constant magnetic permeability. Hence, if we consider the work done, $d\,E$, in raising the currents i and i' by small increments, $d\,i$ and $d\,i'$, in a small time, $d\,t$, we find it consists of four parts—a part, $L\,i\,d\,i$, representing work done by the current i against its own counter-electromotive force, and a similar part, $N\,i'\,d\,i'$, for the other circuit, then a portion, $M\,i\,d\,i'$, representing the work done by the current i in its own circuit against the induced electromotive force due to the increment of the current i' in the other, and lastly, a similar part, $M\,i'\,d\,i$, for the second circuit. Hence, we have

$$d\,E = L\,i\,d\,i + M\,i\,d\,i' + M\,i'\,d\,i + N\,i'\,d\,i'.$$

Integrating this between the limits zero and I for one circuit, and zero and I′ for the other, we find the whole energy represented by the two currents I and I′ flowing in the circuits to be

$$E = \frac{1}{2}\,L\,I^2 + M\,I\,I' + \frac{1}{2}\,N\,I'^2.$$

The electro-kinetic energy is said to be a quadratic function of the currents and the inductances.

§ 9. **Dimensions of the Coefficient of Self-Induction or Inductance.**—Every physical quantity must be measured in units of a like kind or dimension. The height of a steeple cannot be properly expressed in cubic miles or the area of a field in gallons, because the unit is not in these cases of like *dimensions* or kindred nature to the quantity being measured. A quantity of work or energy is always measured by its equivalent in *(force × distance)*, by which is meant a certain force overcome through a certain space. Hence the electro-magnetic energy $\frac{1}{2}\,L\,I^2$ associated with a given current I in a conductor of inductance L must be measurable directly in *ergs* or *foot-pounds*.

The square of a current strength is a quantity of like dimensions with a mechanical force. In a Siemens electrodynamometer, the torsional force of spring measures the square of the current strength going through the instrument.

Since, then, the numerical expression for electro-magnetic energy consists of two factors, one of which is the square of a current strength, and, therefore, of like dimensions to a *force*, the other factor, viz., the inductance L, must be of like dimensions to a *distance*, or be expressible as a length. Inductance is, therefore, measurable in inches or centimetres.

§ 10. **The Unit of Self-Induction or Inductance.**—The absolute unit of inductance is the unit of length. In the practical system of electrical measurement the unit of length is the earth-quadrant or 10^9 centimetres. The practical unit of inductance is, then, 10^9 centimetres. It has been proposed to call this unit of inductance a *Secohm*, in order to possess a single short term for it in similar unitary measurement to the system which contains the Ohm, the Volt, the Ampere, the Coulomb, the Farad, the Watt, and the Joule.

§ 11. **Constant and Variable Inductance.**—It is essential to bear in mind that in the case of circuits consisting of non-magnetic material and immersed in or surrounded wholly by material which is of constant magnetic permeability or magnetic conductance for lines of magnetic force, the inductance of those circuits is a constant quantity depending only on the geometrical form of the circuits. This is the case when the paths of the lines of induction embracing the circuits lie through media, such as air or insulators generally, or diamagnetic materials, such as copper, brass, or wood.

In these cases the magnetic induction through the circuit is always in simple proportion to the magneto-motive force, and for these media we may call the ratio of the line integral of magnetising force to the magnetic induction along that circuit the *magnetic resistance** of that magnetic circuit, or

* Objection may be taken on some grounds to the general use of the term *magnetic resistance*. It has become, however, sanctified by usage. Faraday used freely the term *magnetic conduction* for lines of force. Mr. Oliver Heaviside has proposed the term *magnetic reluctance* as a substitute for magnetic resistance.

$$\text{The magnetic resistance of the magnetic circuit} = \frac{\text{Line integral of magnetic force}}{\text{Induction along the magnetic circuit.}}$$

When, however, we have to deal with the cases most frequent in practice, in which the magnetic circuit is partly or wholly of iron or other para-magnetic metal in which the induction bears no constant ratio to the magnetising force, and in which cases the flux of induction takes place wholly or partly in media of variable permeability, it is necessary to recognise that the inductance of a circuit is no longer a constant quantity. The coefficient of self-induction requires to be more specifically defined for electric circuits embracing magnetic circuits of variable permeability. It may be defined in three ways*—

First, as the ratio between the counter-electromotive force in any circuit and the time rate of variation of the current producing it.

Second, as the ratio between the total induction through the circuit and the current producing it.

Third, as the energy associated with the circuit in the form of magnetic field, due to unit current in that circuit, or as the coefficient by which half the square of the current must be multiplied to obtain the electro-kinetic energy of the circuit at that instant.

These definitions may be algebraically expressed as follows:—

$$\text{(i.)} \quad e = L_1 \frac{di}{dt}$$

$$\text{(ii.)} \quad N = L_2 i \text{ and } e = \frac{d(L_2 i)}{dt}$$

$$\text{(iii.)} \quad T = \tfrac{1}{2} L_3 i^2.$$

If the magnetic circuit is wholly composed of material of constant magnetic permeability, then L_1, L_2 and L_3 are identical, but in the case of iron magnetic circuits, or circuits partly or wholly of ferro-magnetic material, they have not the same

* *See* Mr. W. E. Sumpner, "The Variation of Coefficients of Induction," *Phil. Mag.*, June, 1887, p. 453.

value. These coefficients are all closely related to the curve of magnetisation.

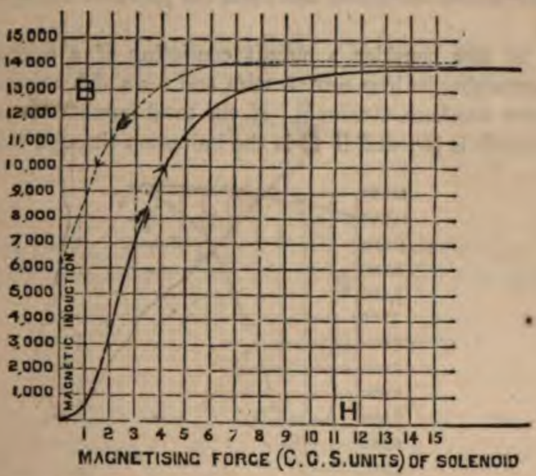
§ **12. Curves of Magnetisation.**—A long iron wire or slender rod, of which the length is at least 200 times the diameter, is wound over with a layer of insulated wire, forming a solenoid of many turns. If the number of turns of the solenoid per unit of length is n, and this solenoid is traversed by a current of A amperes, the magnetic force, or the strength of the field due to the solenoid at the centre of the helix, is $4\pi n \frac{A}{10}$ C.-G.-S. units. If the diameter of the rod is d, the cross section of the solenoid will be nearly $\pi \frac{d^2}{4}$, assuming the iron wire to fill up the interior of the solenoid, and the number of lines of force, or the induction through the centre of the helix, if the iron is removed, will be

$$\pi \frac{d^2}{4} \quad 4\pi n \frac{A}{10} = \frac{\pi^2}{10} n d^2 A = \cdot 9869\, n\, d^2 A$$

in C.-G.-S. units.

Accordingly, a measurement of A, or the ampere magnetising current, enables the magnetising force of the helix to be calculated. Let the iron wire be now placed in the helix. Its presence enormously increases the number of lines of force through that space. If the centre of the helix is embraced by a small coil of wire in circuit with a calibrated ballistic galvanometer and a current is started in the helix, this embracing coil has a certain number of lines of induction suddenly put into it, the number of which becomes known when the constant of the ballistic galvanometer is determined. By increasing the magnetising current suddenly by steps or jumps, and observing in each case the "throw" of the ballistic galvanometer, a curve can be prepared of which the horizontal abscissæ represent magnetising forces (proportional to magnetising currents), and the vertical ordinates the induction or number of lines of induction traversing the central portion of the long iron wire. These curves are called *curves of magnetisation*. In Fig. 19 are represented the curves of magnetisation of an iron wire 200 times as long as it is thick, taken both for ascending (firm line) and descending (dotted line) magnetising forces.*

Denoting as before by **B** the total induction through the median cross section of the iron, and **H** the magnetising force (proportional to A, the magnetising ampere current), we can describe these curves shortly as the (**B**, **H**) curves for iron. It is found that the upward curve indicating the relation of **B** to



Curve of Magnetisation of Iron Rod. Length = 200 diameters (Ewing).

FIG. 19.

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* This curve is taken from Prof. J. A. Ewing's paper, "Experimental Researches in Magnetism," *Trans. Royal Soc.*, Part II., 1885, p. 535. This elaborate paper is an important contribution to the subject.

See also Dr. J. Hopkinson, "Magnetisation of Iron," *Trans. Royal Soc.*, Part II., 1885. p. 455.

The most complete summary of recent research in magnetism is to be found in Prof. Chrystal's article, "Magnetism," in the *Encyclopædia Britannica*, Ninth Edition.

See also "On the Lifting Power of Electro-Magnets and Magnetisation of Iron," Shelford Bidwell, *Proc. Royal Soc.*, June 10, 1886.

Other references to valuable papers on the magnetisation of iron are—

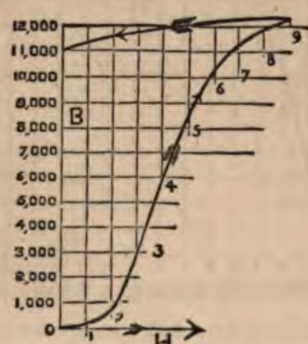
Lord Rayleigh "On the Energy of Magnetised Iron," *Phil. Mag.*, August, 1886, p. 175.

Lord Rayleigh "On the Behaviour of Iron and Steel under the Operation of Feeble Magnetic Forces," *Phil. Mag.*, March, 1887, p. 225.

Ewing and Low "On the Magnetisation of Iron in Strong Fields," *Proc. Royal Soc.*, March 24, 1887, Vol. XLII., p. 200.

H for increasing currents lies below that indicating the same for decreasing currents. The ratio of **B** to **H** is the permeability (μ) of the iron. Hence, if from the magnetising curve another curve is obtained, the horizontal ordinates of which are the various values of **B**, and the vertical ones the corresponding ratio of **B** to **H** or μ, we obtain the permeability curve for that particular sample of iron. The permeability depends not only upon the actual induction, but upon the previous history of the iron.

Let us now consider a circuit consisting of a solenoid or helix containing its iron core or wire, and for which a magnetisation curve has been obtained. If the total number of turns of the solenoid is N^1, and if **B** is the induction through the core.



Curve of Magnetisation of Soft Iron Ring (Ewing).

FIG. 19A.

then if the solenoid is either very long or is bent round so as to form a circular or endless helix, it is easily seen that the circuit is traversed by N^1 **B** lines of induction. This quantity is therefore the electro-magnetic momentum of the circuit, corresponding to a current of A amperes flowing through it, and it represents, as before explained, the total number of lines of induction, N, which penetrate through or are embraced by the circuit. Suppose, then, at any instant when a varying current is passing through the electro-magnet the current strength has a value i, and the induction a value N. The ratio $\frac{N}{i} = L_2$ gives us the inductance of the circuit according to the second method of measurement of the coefficient, and from the first mode of measurement we have $e = L_1 \frac{d\,i}{d\,t}$;

but since $e = \frac{d\,N}{d\,t} = L_1 \frac{d\,i}{d\,t}$,

it follows that $L_1 = \frac{d\,N}{d\,i}$.

As, however, N and i are respectively proportional to **B** and **H**, the induction and the magnetising force, we see that

$$L_1 \propto \frac{d\,\mathbf{B}}{d\,\mathbf{H}},$$

and $$L_2 \propto \frac{\mathbf{B}}{\mathbf{H}};$$

in other words, the first coefficient of self-induction L_1, which we may distinguish as the primary coefficient of self-induction

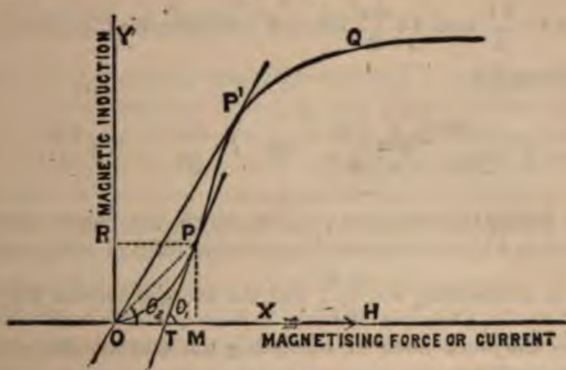


FIG. 20.

is represented by the *slope* of the magnetising curve at any point or the inclination of its tangent to the horizontal axis; whilst L_2, or the secondary coefficient, is represented by the ordinate of the permeability curve corresponding to that particular induction if we assume the magnetic circuit to be homogeneous or all of one material.

Let O P Q (Fig. 20) be a magnetising curve for ascending magnetism. At any point P draw P T a tangent to the curve, and project P on to the axes O X, O Y. Drop perpendiculars P M, P R on O X and O Y, then O M represents the magnetising force or current and O R represents the induction corresponding to the point P. Join P O, then the tangent of the angle P T M represents $\frac{dN}{di}$ or $\frac{dB}{dH}$, or L_1, and hence $L_1 = \tan$ P T M $= \tan \theta_1$, and the tangent of the angle P O M represents $\frac{N}{i}$, or $\frac{B}{H}$ or L_2, and hence $L_2 = \tan$ P O M $= \tan \theta_2$. We see that L_1 and L_2 are identical near the origin, and also at another point P^1 further up on the curve L_1 and L_2 are the same, but that between O and P^1, L_1, becomes greater than L_2, and between P^1 and infinity L_2 is greater that L_1. We have then to see how the magnetising curve gives us L_3 or the tertiary coefficient. In any circuit let the current at a time, t, be i, and let it embrace N lines of induction due to itself; at times $t-\frac{\delta t}{2}$ and $t+\frac{\delta t}{2}$ let the corresponding currents and inductions be

$$i-\frac{\delta i}{2}, \quad N-\frac{\delta N}{2}, \quad \text{and} \quad i+\frac{\delta i}{2}, \quad N+\frac{\delta N}{2}.$$

Then during the short time, δt, in which the increment of in duction is δN, the counter-electromotive force of self-induction will be numerically $e = \frac{dN}{dt}$, and the average current will be i. Hence the work done in increasing the current during that interval will be

$$e\,i\,dt = \frac{dN}{dt}\,i\,dt = i\,dN.$$

And hence the whole work done against self-induction in bringing up the current from zero value to a final value I is

$$\int_0^I i\,dN.$$

But this work is, as we have seen, represented by $\frac{1}{2}L_3 I^2$; hence

$$\tfrac{1}{2} L_3 i^2 = \int_0^i i\,d\mathrm{N},$$

$$\text{or } L_3 = \frac{2\int i\,d\mathrm{N}}{i^2} \propto \frac{\int \mathsf{H}\,d\mathsf{B}}{\mathsf{H}^2}.$$

Referring again to Fig. 20, it is seen that the area contained between the lines O R, R P, and the portion of the magnetisation curve O P is represented by $\int \mathsf{H}\,d\mathsf{B}$, and hence the ratio of this area to the area of the square on O M is the value of L_3.

The general relation of L_3 to L_2 and L_1 cannot be determined without knowing that between N and i or **B** and **H**.

Since $\qquad \mathrm{N} = L_2 i$ and $d\mathrm{N} = L_1\,d i$,

it follows that $\qquad L_1 = L_2 + i\dfrac{d L_2}{d i}$,

which gives a general relation connecting L_1 and L_2. We can sum up the foregoing by saying that when the lines of magnetic induction flow in a medium, of which the magnetic permeability is constant, such as air, the coefficient of self-induction is a constant quantity determined by the form of the circuit; but when the paths of the lines of induction lie wholly or partly in media, such as iron, of which the permeability is a function of the induction, or varies with the induction, then the coefficient of self-induction is variable and may be defined in three ways, the numerical value for a given circuit not being capable of being stated generally, but only specified for each different value of the induction, and can be represented best graphically. If, however, the relation of $\frac{\mathsf{B}}{\mathsf{H}}$ could be expressed algebraically in terms of **H**—that is to say, if the equation to the permeability curve were known—then L_1, L_2 and L_3 could be expressed algebraically also in terms of magnetising force or magnetising current.

In the absence of such knowledge the value of the variable coefficients of self-induction of electric circuits embracing magnetic circuits wholly or partly of ferro-magnetic material can only be expressed as a general algebraical function of the magnetising current or in the form

$$L_i = P + Q\,i + R\,i^2 + \&c.,$$

where P, Q, R, &c., are constant quantities and i is the value of the current to which corresponds a value of the inductance equal to L_i.

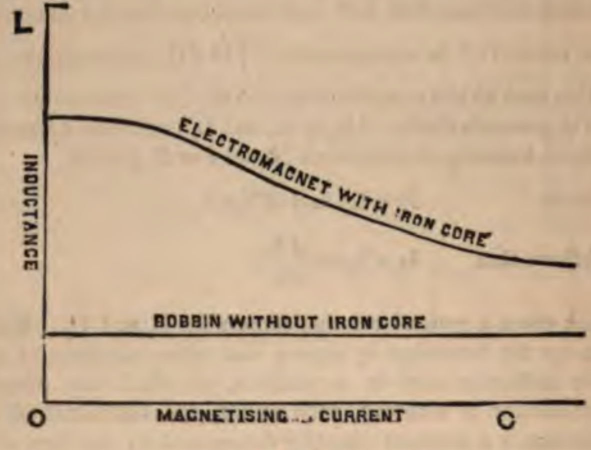


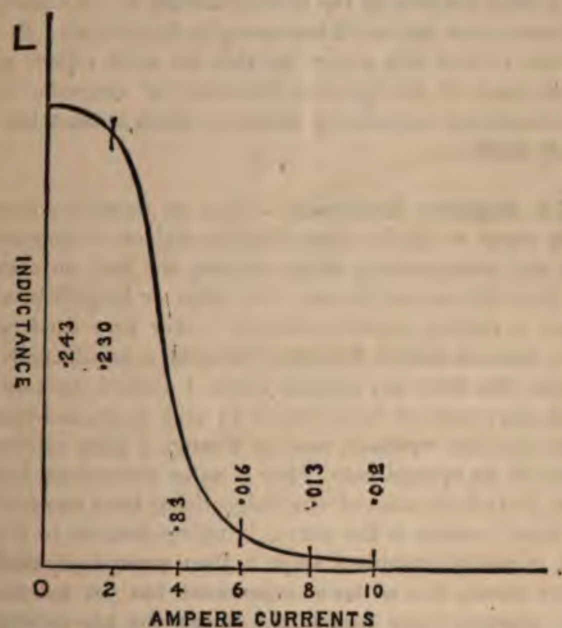
Fig. 21.

§ 13. **Graphical Representation of Variation of Coefficient of Inductance.**—If an electro-magnet consisting of an iron bar or ring having on it magnetising coils have the inductance (L_2) of its circuit determined for various values of its magnetising currents, we find that as the magnetising current is taken upwards from a very small value this inductance at first increases and then decreases. Starting from a point of about half magnetic saturation and continuing to press the induction upwards, the varying inductance will be represented approximately by a curve such as is shown in Fig. 21.* This, it is

* Taken from a paper by M. Ledeboer, presented by M. Lippmann to the Academie des Sciences, June 15, 1886. See *La Lumiere Electrique*, Vol. XX., p. 600, 1886.

easily seen, must be the case from the relation of the inductance L_2 to the general form of the magnetisation curve of iron.

In Fig. 22 is given also a curve showing the variation of the inductance corresponding to various inductions through the iron core of an induction transformer, the inductances measured being those of the primary coil when certain currents represented by the horizontal ordinates created these inductions in the core by traversing the secondary circuit.*



Curve of Self-Induction of Primary Circuit of Transformer with Iron Core.

FIG. 22.

Referring once more to the magnetisation curve of iron given in Fig. 19, it is there seen that the curve consists of two branches; one of which represents the relation of B to H for *ascending* and the other for *descending* magnetisation. The de-

* The values of L_2 and A used for plotting this curve are taken from a paper by Mr. W. E. Sumpner "On the Variation of the Coefficients of Induction," *Phil. Mag.*, June, 1887, p. 468.

scending branch lies above the other, and has at corresponding abscissæ generally a less slope. Hence it follows that the coefficients of induction measured in the first (L_1) or second manner (L_2) will differ not only with the induction but with the direction of the magnetising force whether ascending or descending. When a periodic electromotive force (*see* Chap. III.) is applied to an inductive circuit with ferro-magnetic circuit the coefficient of induction will not only be variable but two-valued; it will depend not only on the current flowing in the circuit, but on whether that current is increasing or diminishing. In order to make evident the reason for this we must return to the consideration of the peculiar behaviour of magnetic metals under changing magnetising forces, to which allusion has been already made.

§ **14. Magnetic Hysteresis.**—When an ascending magnetisation curve is drawn connecting the values of magnetising force and corresponding magnetisation, we find on examination that the curves for soft iron rings or longish iron bars present a certain general similarity. For very small values of the force the curve begins by rising at a small angle. By the time the force has reached about 1 C.-G.-S. unit or so a pretty sharp upward bend begins to take place, and then the curve continues upwards, passing through a point of contrary flexure in its upward path. For a value somewhere between 5 and 10 C.-G.-S. units of the magnetising force another bend or "knee" occurs in the curve, bringing it down to a lesser slope, at which diminished slope it then proceeds upwards for further forces, but as far as experiment has yet indicated it never becomes quite horizontal. When the bar or ring has reached the state popularly called saturation, let the magnetising force be gradually diminished. It is found that the magnetisation now has, during the descent, for the same force, a higher value than during the ascent. If at any point in the descent we pause and begin again to increase the magnetising force, and then after a certain rise decrease it again to the point at which we first arrested the descent, these values of the magnetisations exhibit *a loop* on the magnetisation curve (Fig. 23). If we operate on a ring, and draw the complete magnetisation curve described in carrying the ring from strong magnetisation in one direction to strong magnetisation in the other, we find the curve has two branches (Fig. 24) enclosing a large area. If the horizontal distances or abscissæ represent magnetising force **H**, and corresponding vertical ones the magnetisation **I**, then any area on the plane of **H**, **I**, is analytically represented by the integral

$$\int \mathsf{I}\, d\, \mathsf{H}.$$

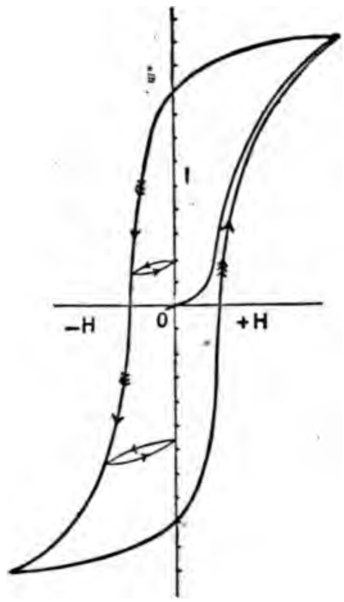


Fig. 23.

Magnetisation curve for very soft iron ring, showing loops due to hysteresis on the descending branch, and enclosed area due to a complete cycle (Ewing).

The physical meaning of these loops or enclosed areas in the magnetisation curves seems first to have been pointed out by Prof. E. Warburg.* This "lagging behind" of the magnetisation has been called by Prof. Ewing *magnetic hysteresis.*

The quantity $-\int \mathsf{I}\, d\, \mathsf{H}$, or the area of any loop of a magnetising curve formed by taking the metal through a complete

* Wiedemann's *Annalen*, XIII., p. 141, 1881.

magnetic cycle, represents in absolute measure the energy spent per unit of volume in performing this cycle. Thus, in Fig. 24 the whole area included between the upward and down ward curve represents the work spent per unit of volume of the ring in taking the ring from strong positive to strong negative magnetisation and back again.

The curve in Fig. 24, taken from a paper by Prof. Ewing, represents the course of the magnetisation in conducting the ring

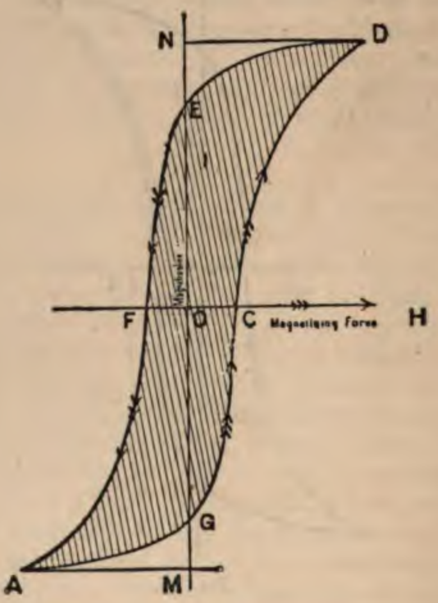


Fig. 24.

Complete Magnetisation Curve for soft iron ring (Ewing) carried from strong positive to strong negative magnetisation. The arrows show the direction of the magnetising operation, the shaded area the work done due to hysteresis.

from strong positive magnetisation O C to equally strong negative O F. The "residual magnetism" or "retentiveness" is represented by O E, and may amount to as much as 93 per cent. of the maximum magnetisation. The work spent in carrying the iron round the magnetic cycle or along the course C D E F A G C is represented by the integral $-\int \mathsf{I}\, d\mathsf{H}$, and is represented numerically per unit of volume by the area (shaded) enclosed by the two curves. It has been shown by Maxwell* that the energy represented by a certain magnetic induction, B, produced by a certain magnetising force, H, is numerically equal per unit of volume of the material to $-\frac{1}{8\pi}$ times the product of the magnetic induction and the magnetic force resolved in the same direction. In the case before us B and H have the same direction; hence, differentiating this expression for the energy per unit of volume, we have the expression

$$dE = -\frac{1}{8\pi}(H\,dB + B\,dH),$$

which denotes the value of the increment of energy per unit of volume expended in increasing B by dB and H by dH.

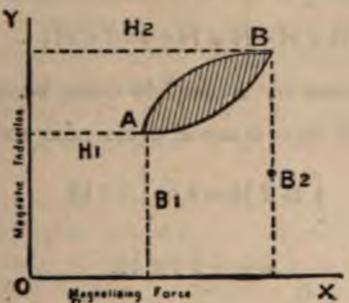


FIG. 25.

When the change is cyclic

$$\int H\,dB = \int B\,dH,$$

for if we are integrating the area of a closed loop (Fig. 25), A B, we arrive at the same result, viz., the area of the loop, whether we proceed by integration along O X, which is obtaining the value of

$$\int_{H_1}^{H_2} B\,dH,$$

* "Electricity and Magnetism," Vol. II., § 636.

or proceed by integrating along O Y and obtain

$$\int_{B_1}^{B_2} H\,d\,B.$$

Hence, integrating the equation

$$d\,E = -\frac{1}{8\pi}(H\,d\,B + B\,d\,H),$$

for a complete cycle of magnetisation gives us

$$E = -\frac{1}{4\pi}\int B\,d\,H;$$

but by the fundamental equation

$$B = H + 4\pi I,$$

and hence $\quad B\,d\,H = H\,d\,H + 4\pi I\,d\,H;$

but $\int H\,d\,H$ is zero for a complete cycle, because for each value of H or d H there is one of equal value and of opposite sign; hence $\quad \int B\,d\,H = 4\pi \int I\,d\,H$

and $\quad E = -\int I\,d\,H.$

In his paper entitled "Researches in Magnetism" (*Phil. Trans.*, Part II., 1885), Prof. Ewing has given the values of the energy dissipated in ergs per cubic centimetre experimentally determined for complete magnetic cycles performed on various samples of iron :—

Sample of iron operated upon.	Energy dissipated in ergs per cubic centimetre during a complete cycle of doubly-reversed strong magnetisation.
Very soft annealed iron	9,300 ergs.
Less soft annealed iron	16,300 ,,
Hard drawn steel wire	60,000 ,,
Annealed steel wire	70,500 ,,
Same steel, glass hard	76,000 ,,
Pianoforte steel wire, normal temper..	116,000 ,,
Same, annealed	94,000 ,,
Same, glass hard	117,000 ,,

From the above we can deduce that, roughly speaking, it requires 28 foot-pounds of energy to make a double-reversal of strong magnetisation in a cubic foot of soft iron. The energy so expended can take no other form than that of heat diffused throughout the mass.

A similar table of experimental results has been given by Dr. J. Hopkinson (*Trans.* Roy. Soc., Part II., 1885, p. 463), in which Paper the chemical analysis of the samples operated upon is given. The highest value of specific hysteresial dissipation was found for Tungsten steel, oil hardened, in which the value of the energy in ergs per cubic centimetre dissipated in a complete magnetic reversal was 216,864.

This *static hysteresis* is a quality of iron in virtue of which reversal of magnetisation is accompanied by dissipation of energy. If we fall back upon current hypotheses of magnetism (Ampere-Weber-Maxwell) to afford us some explanation of it, we can find it to some extent by supposing that the magnetic molecules or molecular magnets, the arrangement of which constitutes magnetisation, move stiffly, and the dissipation of energy we are considering is the work done in making the necessary magnetic displacement against a sort of internal magnetic friction.

This dissipation of energy into heat during magnetisation is something quite apart from any production of heat by eddy (or so-called Foucault) electric currents induced in the mass, and would take place in iron so perfectly divided that no eddy currents could exist.

One result of Prof. Ewing's researches has been to show that if the iron is kept in a state of mechanical vibration hysteresis is greatly diminished, and the value of the energy dissipated in a complete cycle is much reduced. The removal of strong residual magnetism from soft iron by slight tapping or twisting has also been noticed and commented on by Prof. Hughes.*

If we place the above facts side by side we see that in magnetising the field magnets of an ordinary continuous-current dynamo, work has to be done to a small amount at first to bestow upon the iron its initial magnetisation. If iron possessed the property of retaining strongly some 90 per cent.

* Prof. D. E. Hughes "On the Cause of Evident Magnetism in Iron," *Proc. Soc. Tel. Engineers*, May 24, 1883, p. 3.

of its maximum magnetisation when the magnetising force is withdrawn there would be no further need for a field-exciting current; but as the least shock or vibration removes the slightly held magnetisation of iron, we see that one function of the field-exciting current, and of the energy expenditure necessary to maintain it, is to keep on reproducing the magnetism which the continual vibration knocks out of the iron. Hence, in an ordinary dynamo, we may consider that one part of the duty of the field-exciting current is to keep on re-generating magnetisation because mechanical vibration is continuously tending to remove it.

Hysteresis will also operate to cause a dissipation of energy in the armature of dynamos. For in this case we have a mass of soft iron, viz., the armature core, which has its direction of magnetisation reversed every revolution. Suppose the core has a volume of 9,000 cubic centimetres, and that it makes 15 revolutions per second. Taking the specific hysteresis for this sample of iron at 13,356 ergs, we find that the dissipation of energy in ergs per second is equal to $9{,}000 \times 15 \times 13{,}356 = 180 \times 10^7 = 180$ joules, or a loss of about a quarter of a horse-power.

Experiments have been made by Joule and others to determine by direct observation the heating effect of magnetisation upon iron, but in these direct experiments it is probable that the results were mostly impure, and qualified largely by the production of heat by local currents.*

Since about 10,000 ergs per cubic centimetre are dissipated by a double reversal of strong magnetisation of soft iron, it is not difficult to show that the consequent rise of temperature, if all the heat is retained in the iron, is ·000284° C., or that some 4,000 reversals would be required to raise the temperature 1° C., even provided all the heat generated is retained in the metal. Hence it follows that any very marked rise in temperature cannot be due to hysteresis.

In connection with this subject it is interesting to touch upon the question whether magnetised iron *per se* represents

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* See Joule's "Scientific Papers," Vol. I, p. 123, on the Calorific Effects of Magneto-Electricity, and on the Mechanical Value of Heat. Also *Phil. Mag.*, series 3, Vol. XXIII., p. 263.

a store of energy; in other words, is magnetised iron analogous to a bent spring?

It is not disputed that available energy accompanies the magnetisation of a short iron bar, but this is a virtue of free polarity. In consequence of the attraction of opposite poles, if a magnetic bar were flexible the two poles would bend round to meet each other, and when this operation was finished there would be no longer any work to be got out of the bar.

In the case, however, of a ring, the magnetisation, though strong, represents no available energy. It requires energy to be applied to demagnetise the iron, and for all practical purposes magnetised iron *per se* cannot be regarded as a source of energy.* It requires work to be done to displace the magnetic particles from their positions, and work also to be done to push them back.

The operation of magnetising iron is not analogous to the stretching of a spring, but rather to the bending of an inelastic wire, such as a lead wire, in which work has to be done in twisting the wire to make the several sections shear over each other, though the operation when finished represents no available work.

In addition to the *static hysteresis*, experiments by Prof. Ewing and others also indicate the existence of *viscous hysteresis*. There is evidence of a true *time lag* in magnetisation, especially in the earlier stages of magnetisation. The iron requires time for a given magnetising force to produce its effect, and this sluggishness does not appear explicable as an effect either of induction in the iron or to self-induction in the magnetising circuit. The result of this magnetic viscosity is to augment the value of $-\int \mathsf{I}\, d\mathsf{H}$ in any rapidly performed cycle, and, consequently, to increase the dissipation of energy into heat. When we come to consider the question of induction transformers it will be seen that this viscous hysteresis has to be taken into account in establishing the full theory.

With respect to this *time lag* of magnetisation Prof. Ewing says:† "I repeatedly observed that when the magnetising

* Lord Rayleigh on "The Energy of Magnetised Iron," *Phil. Mag.*, August, 1886.

† *Phil. Trans.*, Part II., 1885, p. 569.

current was applied to long wires of soft iron there was a distinct *creeping up* of the magnetometer deflection after the current had attained a steady value."*

If this is so it would seem that a sufficiently rapid reversal of magnetising force would render such force inoperative in magnetising the iron. Just as the self-induction of a circuit operates to make a certain delay in the appearance of the current corresponding to the application of a given electromotive force, and hence renders a circuit apparently a worse conductor for rapidly reversed electromotive forces, so the magnetic lag would introduce a sort of *magnetic self-induction* or retardation in the appearance of the magnetisation corresponding to a given magnetising force.

This time lag appears to be most manifest in the softest iron, and to be especially noticeable near the beginning of the steep part of the magnetisation curve.

In an investigation on the magnetisation of iron under feeble magnetic forces, Lord Rayleigh has also drawn attention to the fact that the settling down of iron when very soft or annealed into a new magnetic state is far from instantaneous.† If the strength of the earth's horizontal magnetic field is called h, Lord Rayleigh has shown that for unannealed iron and steel magnetising forces ranging from $\frac{1}{3}h$ to $\frac{1}{10000}h$ call forth proportional magnetisation. In other words, the susceptibility is constant over this range, and the value of the corresponding permeability is from 90 to 100, and that this small proportional magnetisation takes place independently of what may be the actual magnetisation of the iron provided it is not very near the condition usually called saturation.

The moment, however, that the magnetising force is pushed beyond these limits the phenomena of hysteresis and retentiveness make their appearance. According to Prof. Ewing (*loc. cit.*) the following group or constellation of hypotheses has to be made in order to approximate to a mechanical explanation of the magnetisation of iron :—

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* See also Mr. T. Blakesley "On Magnetic Lag," *Phil. Mag.*, July, 1888, p. 34, in which it is shown that, by means of three dynamometers, the lag due to viscous hysteresis may be measured. Further reference will be made to these experiments.

† Lord Rayleigh "On the Behaviour of Iron and Steel under the Operation of Feeble Magnetic Forces," *Phil. Mag.*, March, 1887, p. 225.

1. An elastic tendency on the part of the magnetic molecules to recover their primitive position when displaced.

2. A static frictional resistance to their displacement and to their return removable largely by vibration.

3. A limit for each such that if the displacement of the molecule exceeds it a permanent displacement not removable by vibration results.

4. Probably a viscous resistance to the displacement and return of the molecules.

5. An unequal distribution amongst the molecules of the frictional resistance, such that in some of the molecules it is, perhaps, very small.

Aided by these suppositions we might proceed to build up, in imagination, a mechanical model, which should imitate, under the application of certain stresses, the behaviour of iron under magnetising force, but it is certain that the result would not be practically of much value; and it must still be left to future research to collect and analyse the necessary facts before the mystery of the magnetisation of iron is fully unraveled.

Before entering into further discussion of magnetic induction between circuits of variable inductance it will be advantageous to consider some problems on current flow in circuits of constant inductance, and to make these easier questions the stepping stones to more difficult ones involved in practical apparatus in which the magnetic circuits are iron circuits, and the inductances of the electric circuits therefore variable and non-constant.

CHAPTER III.

THE THEORY OF SIMPLE PERIODIC CURRENTS.

§ 1. **Variable and Steady Flow.**—In considering the motion, either of actual fluids or of electric currents, we can distinguish two states—the *variable* and the *steady* condition. In the former case the strength or direction of the electric currents or of the fluid velocity is changing at every instant; in the latter case, the flow has settled down into a permanent state. The questions involved in dealing with the variable state present rather more difficulties than do problems in steady flow, for the reason that the notions of *time* and *inertia* enter into these in a way in which they do not when that flow has reached a steady condition. We shall proceed to examine in an elementary manner some features of electrical flow when variable or periodic. We must, however, prepare the way by considering some purely geometrical properties of certain curves, and also some modes of motion which have special reference to the kind of electric current to be considered subsequently. When a mass of water is in motion, a particle of water selected for examination has at any instant a certain velocity in a certain direction. This may be represented graphically by a straight line drawn from that particle representing its velocity in direction and magnitude. Similarly, if electricity is flowing through the mass of a conductor in any manner, it is possible at any point to draw a *vector* or line representing at that instant the direction and magnitude of the current at the point from which the line is drawn. Lines drawn within the mass of a fluid at any points such that the flow at that instant is along or tangential to these lines are called *flow lines.* In the first place, let us make the supposition that the flow has reached a steady condition. The flow lines are then fixed. When this is the case each line of flow becomes the actual path of a fluid particle, and is called a *stream line.* A surface may be supposed to be described in the mass of the fluid everywhere perpendicular or orthogonal to the stream lines; such a surface is called an equipotential or level surface. We may also suppose such a level surface drawn in the mass of a conductor through which a current is flowing. Let any area be drawn on the equipotential surface, and let it be divided up into units of area. If the quantity of fluid or of electricity flowing through each unit of area is the same, and if, moreover, it is the same for each unit during each succeeding instant of time, the current is said to be *steady* and to be uniformly distributed. The quantity flowing per unit of time through any area is the numerical measure of the mean strength of current over that section of the conductor, and the quantity flowing per unit of time through a unit of area is the measure of the mean density of current over that unit of area. If the distribution of current and strength is not uniform, we can only express them at any time and place by calling to aid the language of the differential calculus. If ds be a small area described on an equipotential surface, and if dq be the quantity of electricity which flows in a small time dt through that area ds, and if i is the strength of the current at the centre of that small area at any instant, then in the limit

$$i=\frac{dq}{dt}.$$

§ 2. **Current Curves.**—To fix our ideas, let us now suppose the electric flow to take place through a thin cylindrical conductor, such as a wire, in which at positions sufficiently remote from the ends the stream lines will be parallel to the axis of the wire and the equipotential surfaces perpendicular to it. Consider any one section, and let the flow across this section be variable both in strength and direction—that is to say, let it vary in the quantity of electricity which flows across that section in each succeeding instant, and let the flow be first one way and then the other, varying in any manner, however irregular. We can represent graphically the state of things as regards electric flow at that section by means of a *current curve*. Take a horizontal line (Fig. 1) to represent the uniform flow of time. At successive instants let ordinates be drawn to this line, representing the strength of current flowing past that section, and let them be drawn above (+) or below (−), according as the direction of the flow is to the right or to the left. Thus, if time begins to reckon from O, after the lapse of a time O T the current is positive, and is represented by a line T I. After the lapse of a time O T′, the current is negative, and is represented in strength by a line T′ I′.

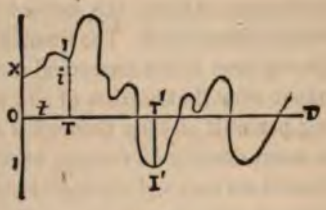

Fig. 1.

This current curve is obviously a single valued function—that is to say, corresponding to a given instant of time the current can only have one value. The curve can never cut itself or double back.

It is necessary to bear in mind the distinction between single and multiple valued functions. A single valued function is one which, when represented graphically by a continuous curve, presents only one value of the ordinate for each value of the abscissa.

In Fig. 2 is represented graphically a single valued function, having only one value of the ordinate $x\,y$ corresponding to a given value of abscissa $O\,x$. In Fig. 3 is represented a curve such that there are five different values of the ordinate of the curve corresponding to one value of the abscissa $O\,x$. This curve represents a multiple valued function.

§ 3. **Simple and Complex Harmonic Motion.**—Jean Baptiste Joseph Fourier was the first who enunciated the remarkable mathematical theorem now known as "Fourier's theorem."* Generally stated it is as follows:—Any single valued periodic function can be expressed analytically as the sum of a series of terms, the first of which is an arbitrary constant, and each of the following terms is the sine or cosine of an angle multi-

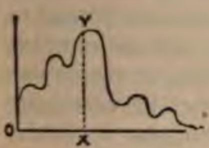

FIG. 2.

Single valued function.

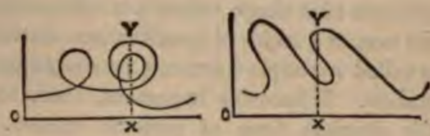

FIG. 3.

Multiple valued functions.

plied by a constant. In other words, this amounts to saying that the equation to any such curve can be found in the form of an infinite series of terms, each one of which is the sine or cosine of an angle multiplied by a constant term. The most remarkable part of Fourier's method is its power even to represent in this form the equation of lines made up of pieces of straight

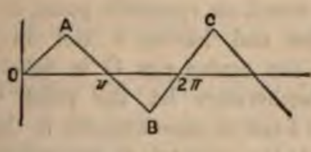

FIG. 4.

lines. Take the case of a line O A B C, &c. (Fig. 4), made up of lines inclined at angles of 60deg., like the teeth of a saw, the length of each line being 2π units of length. It can be shown

* Fourier's great work, "La Théorie Analytique de la Chaleur," was published in 1822, and was an epoch making work in mathematical science.

that the equation to the broken line O A B C, &c., is the series

$$\frac{4}{\pi}\left\{ \sin x - \frac{1}{9} \sin 3x + \frac{1}{25} \sin 5x - \&c. \right\}$$

or is an expression formed of the sum of a series of terms, each one of which is the sine of an angle multiplied by a constant term. Physically interpreted, Fourier's theorem means that any variation of motion, which can be represented by the changing ordinate of a single valued periodic curve, can be expressed as the sum of a series of simultaneous motions, each one of which is called a simple harmonic or simple periodic, or simple sine motion. It becomes important, then, to start by examining the simplest form of periodic motion. Suppose a circular disc (Fig. 5), having a pin at its centre, O, to be pivoted

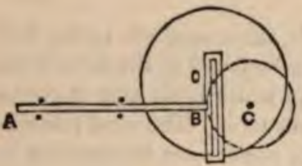

FIG. 5.

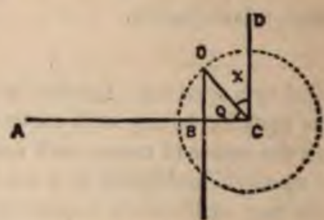

FIG. 6.

so as to revolve round an excentric point, C. Let a T bar, moving in guides and having a slot in the cross-piece, be so fixed that the centre pin O is constrained to move in the slot. Furthermore let the point C round which the disc moves be fixed to some support in the line of the bar A B produced. If the excentric is compelled to move round C, the extremity of the bar A will move backwards and forwards with a motion called a simple harmonic motion or a simple periodic motion.

For it is clear the point O (Fig. 6) is compelled to move in a circle round C as a centre, and hence the distance of the point A from C at any instant is the length of the bar A B *plus* the length B C, which is the projection of O C on the line A C. The point B therefore executes a simple vibration to and fro along the line A C as O moves round, and the point A imitates the motion of B. If the angle O C D is called x and the radius O C is a, then the length B C is $a \sin x$, and the displacement of A at any instant from its mean or middle position has the same value. The motion of A is called a simple harmonic motion, and the above excentric and T bar is a mechanical device for compelling a point to describe a simple harmonic motion (abbreviated into S. H. M.). If such an harmonic motion be executed by point A (Fig. 7) whilst at the same time a strip of paper, S S′, is caused to move uniformly in a direction perpendicularly to the line A B, a tracing point fixed to A will describe on the paper a curve of which the ordinate A Y is proportional to the sine of the abscissa X Y, or the equation to the curve will be of the form $y = a \sin x$, a being some constant

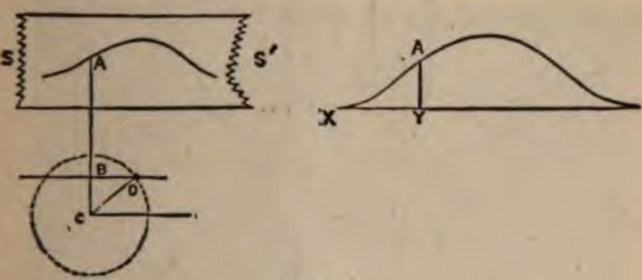

Fig. 7.

quantity. Hence a simple periodic curve is also called a *sine* curve.

By combining together two similar pieces of mechanism it is possible to construct a machine which can add together graphically two simple harmonic motions in the same line, but of which the phase angles x and the amplitudes a are different. Machines for doing this have been devised by Sir W. Thomson and also by Mr. Stroh. Apart from complications the general principle is as follows :—

Let a cord pass over four pulleys (Fig. 8), two of which, F_1 F_2, are fixed in space, and two, M_1 M_2, can be made to rise and fall in vertical lines with a simple harmonic motion by being attached to T bars and excentrics. If the cord has one end, B, fixed and the other end, A, free, it is easy to see that if either the pulley M_1 or M_2 rises and falls along a vertical line and the cord is just kept tight, the free end A will be displaced by an amount equal to twice the displacement of M_1 or M_2, and as M_1 or M_2 moves up and down with a S.H.M., the free end of A will also execute similar vibrations. If M_1 and M_2 move together the displacement of A at any instant is equal to the sum of the displacements of M_1 and M_2. By providing the end A with a tracing point and moving under it uniformly a sheet of paper in a direction perpendicular

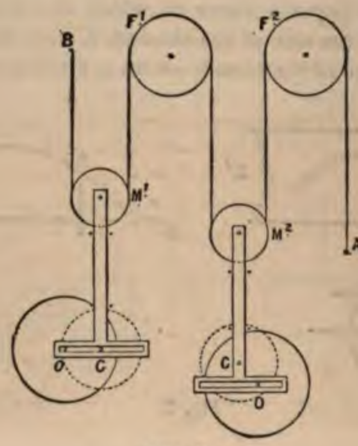

FIG. 8.

to the direction of motion of A, it will describe a curve of which the equation will be of the form

$$y = a \sin x + a' \sin x',$$

a and a' being the amplitudes, and x x' the phase angles of the two motions of M_1 and M_2 respectively. This apparatus, or one of similar principle, has been devised and employed by Sir W. Thomson in his researches on the tides. It will be evident from the foregoing explanation that a machine can be constructed capable of causing a tracing point to move to and fro across a uniformly flowing sheet of paper, with a motion compounded of any number of simple harmonic motions of different amplitude and phase taking place in the same straight line.

As an example, in Fig. 9 are shown two simple sine curves, represented by the firm lines, of which one has double the wave-length and about two and a-quarter times the amplitude of the other. If these curves are superimposed, and a new curve, represented by the dotted line, formed by adding the ordinates X y_1, X y_2, of a common abscissa, O x, into a third, X y_3, then we obtain, by repeating this at all points, a new curve, which may be considered by Fourier's analysis to be resolvable into the simple sine curves. The dotted curve is a complex sine curve, and the two firm-line curves are its two components.

If, then, any single valued function is graphically represented —that is to say, any such curve as in Fig. 1—we see that this

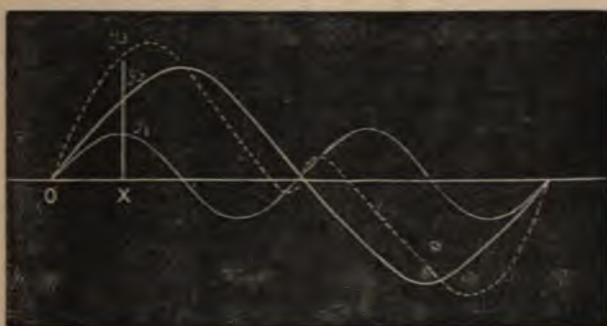

FIG. 9.

curve may be described by a point which moves horizontally with a uniform velocity, whilst at the same time it executes in a vertical direction a movement which is the sum of a number of simple harmonic motions superimposed upon one another. The combination of these two rectangular motions causes the point to describe the curve considered.

In subsequent chapters we shall be examining effects which are due to periodic or fluctuating electric currents. Fourier's theorem gives us, when applied to these cases, a simplification of immense value in that it enables us to see that, however complicated may be the fluctuation of current in a conductor, it can always be resolved into the sum of a series of simultaneous currents varying in a simple manner, and each of which can be graphically represented by a simple harmonic curve. The general consideration of periodic currents must then be preceded by an examination of the elementary theory of electric currents of a periodic character, in which the variation is of the most simple kind.

Fourier's theorem applies also to many other physical phenomena of great importance. In acoustics it shows, for instance,

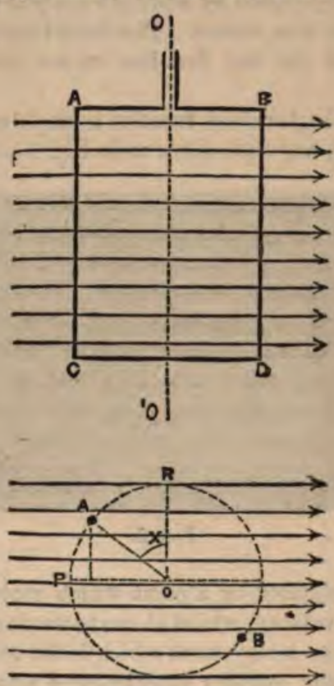

FIG. 10.

that however complicated may be the motion of an air particle in a mass of air through which sound waves are being transmitted, it can be resolved into the sum of a series of motions such as would be produced by the action of tuning forks, each of which gives rise to a motion in the air particles approximately of the nature of a simple harmonic vibration. Helmholtz actually realised this in his synthesis of vowel sounds.

The starting point will be the examination of the most simple possible case of periodic or varying currents. Let A B C D (Fig. 10) be a rectangular frame or conductor, able to revolve round a vertical axis, O O′, in a uniform magnetic field. The adjacent figure represents the same in plan. If the frame revolve round the axis O O′, the total electromotive force acting round the circuit at any instant is numerically equal to the time rate of change of magnetic induction or number of lines of magnetic force passing through the circuit. If H is the field strength, l the length of the side A C, and k the length of the side C D, and x the angle which at any instant the plane of the frame makes with a plane drawn at right angles to the lines of the field, then the magnetic induction or number of lines of force through the frame is the product of H, and the apparent size of the frame, as seen along the direction of the lines of force of the field, or is equal to $\mathrm{H}\, l\, k \cos x$.

If the area of the frame is A, the magnetic induction is $\mathrm{H\,A} \cos x$. The effective electromotive force acting to produce a current in the circuit is numerically equal to the time rate of change (decrease) of the magnetic flux or induction, or to

$$-\frac{d\,(\mathrm{H\,A} \cos x)}{d\,t} = \mathrm{H\,A} \sin x \frac{d\,x}{d\,t}.$$

This last equation is merely a symbolic statement of the fact that if such a frame of area A revolve round an axis perpendicular to the lines of force in a uniform magnetic field, H, with an angular velocity $\frac{dx}{dt}$, then the integral electromotive force acting round the frame at any instant corresponding to an angular displacement x is $\mathrm{H\,A} \frac{dx}{dt} \sin x$.

If the angular velocity remains constant, the effective electromotive force will be simply proportional at any instant to the sine of the angular displacement of the frame from its initial position. Such a frame produces by its uniform revolution a simple sine variation of electromotive force in its own circuit. If we suppose such a frame to have a closed circuit, then this periodically varying electromotive force will produce in the circuit an electric current which varies in strength very nearly as the sine of the angle of the displacement of the frame from its zero position when no lines of force penetrate through its area. Hence, graphically represented, the current varies according to a simple harmonic law, or is a simple sine current. We can then synthesize by the superposition of such simple harmonic electric currents any form of variable current, however complicated. Let a series of such sine inductors be joined up on one circuit (Fig. 11), each capable of being regulated as to angular velocity, and imagine these to revolve in magnetic fields of equal strength. These sine inductors are originally set with the plane of their frames at certain different but fixed angles to the planes at right angles to the fields of force in which they revolve, and they must be supposed to maintain these relative positions during their revolution. Accordingly, the effective electromotive force in the whole

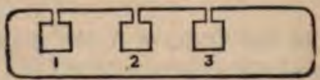

FIG. 11.

circuit, when they are all joined up in series and set revolving at fixed speeds, is represented by a function

$$e = A \sin x + A' \sin x' + A'' \sin x'' + \&c.;$$

and by Fourier's theorem any possible periodic variation of e which, graphically described, is a single valued function can be produced by suitable values of the speeds and phase angles of these sine inductors.

The converse of the above proposition is also true. Let there be any current-generating machine producing in a circuit an electromotive force, and therefore a current varying periodically according to any law. This kind or form of current could be exactly imitated by removing the given machine and substituting a series of sine inductors coupled in series and arranged so as to each produce a simple sine varying E.M.F., the respective sine currents having different phases and amplitudes, but being superimposed upon one another. That is to say, however complicated may be the nature of the periodic current which traverses a circuit, provided the same electric motions are repeated at regular intervals, we may build up this current by suitably superimposing in the same circuit a number of simple periodic currents of certain amplitude and wave-lengths and fixed difference of phase.

The above remarks may be taken as an outline of the analysis of any single valued continuous function into a series of simple harmonic functions. To simplify language we shall in future speak of a curve whose equation is of the form $y = A \sin x$ as a *simple periodic curve*, and if such curve graphically represents the continuous variation of the flow of electricity past any section of a conductor, or the fluctuation of electromotive force in any circuit, we shall speak of such as a *simple periodic current* or a *simple periodic E.M.F.*

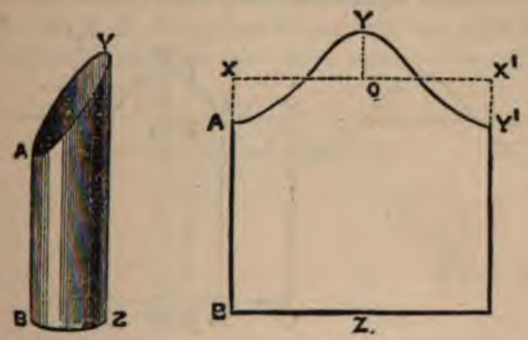

Fig. 12.

Any other mode of variation of these quantities which, graphically represented, would be a single valued curve repeating the same form, will be spoken of as a complex periodic curve, current, or E.M.F., and, by the foregoing analysis, a complex periodic function can be analysed into a sum of simple periodic functions.

§ 4. **Description of a Simple Periodic Curve.**—The following method affords a very easy means of drawing a simple periodic curve. Take a cylinder or tube of pasteboard (*see* Fig. 12) and cut it through obliquely with a sharp knife, taking care to make the cut in one plane. The section of this cylindrical tube by an oblique plane will be an ellipse. Slit the tube open along the line A B and unfold it. Lay it down on another sheet of paper and draw a pencil line guided by the curved edge A Y Y′. Draw a dotted line, X X′, so that its vertical distance below the highest point Y on the curve is equal to its vertical distance above the points A and Y′, or make O Y equal to A X. Then move this cardboard template forward through a distance equal to its own width, and draw another piece of curve repeating the first and similarly placed (*see* Fig. 13).

The resulting curve is a simple periodic or simple sine curve.* The distance X X′, equal to the circumference of the tube or to the width of the template, is called the *wave length*. The distance O Y of the highest point above the mean line is called

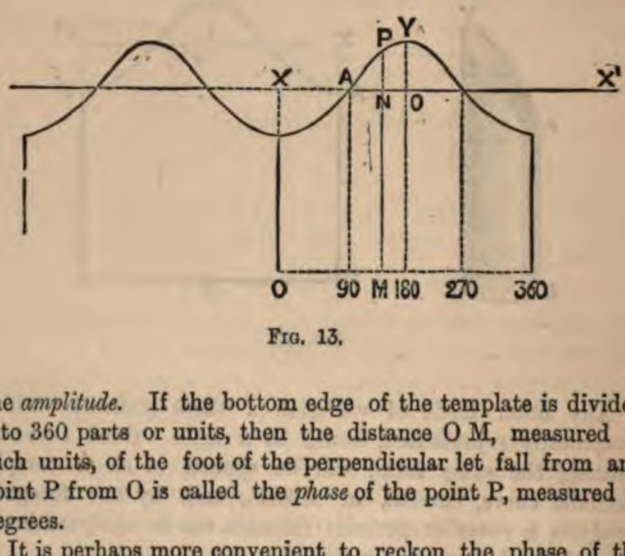

FIG. 13.

the *amplitude*. If the bottom edge of the template is divided into 360 parts or units, then the distance O M, measured in such units, of the foot of the perpendicular let fall from any point P from O is called the *phase* of the point P, measured in degrees.

It is perhaps more convenient to reckon the phase of the point P by the magnitude of the line A N, or the distance of the foot of the perpendicular, let fall from P on X X′ from the point A, where the curve crosses the mean line. The phase of the maximum ordinate O Y is then 90°.

* "Elements of Dynamics" (Clifford), p. 22.

§ 5. **The Value of the Mean Ordinate of a Sine Curve.**— Let Fig. 14 represent the semi-wave of a simple periodic curve; we shall proceed to prove some geometric properties of such a curve. Considering this curve as bounding an area of which the other including line is the datum line X X′, we shall first find the value of the mean ordinate. Let X X′ be divided into equal and very small intervals, such as N N′, of which the length is $d x$, and let X N be called x. Assume as a unit of length the radius of the cylinder, of which X X′ is the semi-circumference. At each of these small elements raise ordinates, such as P N, to touch the curve. We require to find the mean value of all these equi-spaced ordinates when they are infinitely close. The mean value of a number of things is the sum of them divided by their number. If y denote the length of one such ordinate P N, and Σy the sum of all such ordinates when ruled at n equal and exceedingly

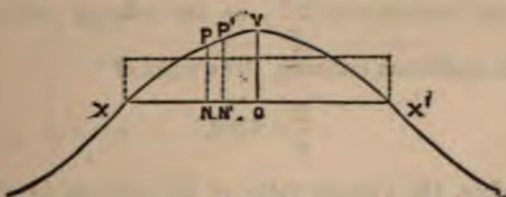

FIG. 14.

small intervals, each of length $d x$, then the average value of these infinitely numerous ordinates is

$$\frac{\Sigma y}{n}, \quad \text{or} \quad \frac{d x \Sigma y}{n d x}, \quad \text{or} \quad \frac{\Sigma y d x}{n d x};$$

but the sum of all such quantities as $y d x$, or P N . N N′ is the sum of all the areas of the little rectangular slips into which these infinitely numerous ordinates divide the area bounded by the curve and X X′, and $n d x$ is the length X X′; hence we have

$$\text{mean ordinate} = \frac{\text{area X Y X}'}{\text{length X X}'}.$$

The area X Y X′ is obtained by integrating the equation to the curve. Calling the maximum ordinate O Y=A, and the distance X N=x, the unit being the radius of the cylinder of which

X X′ is the semi-circumference, we have as the equation to the curve

$$y = A \sin x,$$

and $$\int y\,dx, \quad \text{or } A\int \sin x\,dx$$

between the limits O and π is the value of the area of the curve.

But $$A\int \sin x\,dx = -A \cos x,$$

and this between the limits $x = O$ and $x = \pi$ is equal to 2 A. On the same scale the length

$$XX' = \pi.$$

Hence, the average value of the infinitely numerous and equi-spaced ordinates is $\frac{2A}{\pi}$, or the average ordinate $= \frac{2}{\pi}$ times the maximum ordinate. The value of

$$\frac{2}{\pi} = {\cdot}6369.$$

Therefore, the average value of the ordinate of a simple periodic curve, or the true mean ordinate, is ·6369 of the maximum ordinate, and if a current or an electromotive force varies according to a simple periodic law, the true mean current or the true mean E.M.F. is ·6369 of the maximum current or E.M.F. during the phase.

We have here made use of one simple integration, and it is generally easier to master the elements of the infinitesimal calculus than to construct or follow proofs which aim at avoiding its use. We shall, however, indicate how the value of this mean ordinate may be found from first principles. If we call the length of the base line X X′ l, and divide it into n equal and very small parts of length δx, then $n\,\delta x = l$. Erect at each interval an ordinate whose height is y, then the equation to the curve is $y = A \sin \frac{\pi}{l} x$, where x is, as before, the distance X N. The mean value M of the ordinate is the sum of all the values of the ordinates divided by their number, or is equal to

$$\frac{1}{n}(y_1+y_2+y_3+\&c.)$$

$$\therefore M=\frac{1}{n}A\left\{\sin 0+\sin\frac{\pi}{l}\delta x+\sin\frac{\pi}{l}2\,\delta x+\ldots\right.$$
$$\left.+\sin\left(\frac{\pi}{l}\,\overline{n-1}\,\delta x\right)\right\}$$

The sum of the sine terms in the bracket is known by trigonometry to be equal to

$$\frac{\sin\left(\frac{\overline{n-1}}{2}\frac{\pi}{l}\delta x\right)\sin\frac{n\pi}{2l}\delta x}{\sin\frac{\pi}{l}\frac{\delta x}{2}}.$$

Hence $\quad M=\frac{A}{n}\left\{\frac{\sin\left(\frac{\overline{n-1}}{2}\frac{\pi}{l}\delta x\right)\sin\frac{n}{2}\frac{\pi}{l}\delta x}{\sin\frac{\pi}{l}\frac{\delta x}{2}}\right\}$

which may be otherwise written—

$$M=\frac{A}{n\,\delta x}\frac{l}{\pi}\frac{\frac{\pi\,\delta x}{l}}{\sin\frac{\pi}{l}\frac{\delta x}{2}}\left\{\sin\left(\frac{n\pi\,\delta x}{2l}-\frac{\pi\,\delta x}{2l}\right)\sin\frac{n\pi\,\delta x}{2l}\right\}$$

When n becomes infinite and δx becomes zero, $n\,\delta x$ remains still equal to l; hence the above expression in this case reduces to the following:—

$$M=\frac{A}{\pi}\,.\,2\,.\,\sin^2\frac{\pi}{2}\,.\,=\frac{2}{\pi}A.$$

for the value of $\dfrac{h}{\sin\frac{h}{2}}$ is 2 when h becomes zero.

Accordingly the mean value of the ordinates when they are infinite in number and equi-spaced is $\frac{2}{\pi}$ times the magnitude of the maximum ordinate.

§ 6. **The Value of the Mean of the Square of the Ordinates of a Simple Periodic Curve.**—We require in the next place to find the value of the mean of the square of the ordinates to the same curve, assuming them to be equi-distant and infinite in number. If y_1 y_2, &c., are the ordinates, and n the number, we require to find the value of

$$\frac{1}{n}\left(y_1^2 + y_2^2 + y_3^2 + \&c.\right),$$

the value of any ordinate being, as above,

$$y = \mathrm{A} \sin \frac{\pi}{l} x.$$

If X X′ or l is divided into n intervals, each equal to δx, so that $n \delta x = l$, we have to find the value of

$$\frac{\mathrm{A}^2}{n}\left(\sin^2 0 + \sin^2 \frac{\pi}{l} \delta x + \sin^2 \frac{\pi}{l} 2 \delta x \quad . \quad . \quad . \right.$$

$$\left. + \sin^2 \frac{\pi}{l} \overline{n-1}\, \delta x\right);$$

but, since $$\sin^2 \theta = \frac{1}{2}\left(1 - \cos 2\theta\right),$$

the series in the bracket can be replaced by

$$-\frac{1}{2}\cos 0 + \frac{1}{2} - \frac{1}{2}\cos\frac{\pi}{l} 2 \delta x + \frac{1}{2} - \frac{1}{2}\cos\frac{\pi}{l} 4 \delta x + \&c.$$

$$. \quad . \quad . \quad . \quad . \quad + \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi}{l} 2 n \delta x - \frac{\pi}{l} 2 \delta x\right)$$

for n terms. Hence the mean value M is

$$\mathrm{M} = \frac{\mathrm{A}^2}{n}\frac{n}{2} - \frac{\mathrm{A}^2}{2n}\left(\cos 0 + \cos \frac{\pi}{l}\, 2 \delta x + \&c.,\right)$$

for n terms.

The cosine series forms a progression of terms which begins with unity, since $\cos 0° = 1$ and passes down through zero to -1, and then up from -1 through zero to unity again,

for $$\cos\left(\frac{\pi}{l} 2 n \delta x - \frac{\pi}{l} 2 \delta x\right) = +1,$$

when $n \delta x = l$, and δx becomes infinitely small.

Since the angles are in arithmetic progression we can pick out from this series pairs of cosine terms, such that they are equal in magnitude but opposite in sign, and when taken pair and pair, cancel each other out. The sum of the cosine series in the bracket is thus equal to zero, and, therefore,

$$M = \frac{A^2}{2},$$

that is, the mean of the values of all the ordinates squared taken equi-distant and infinite in number is half the square of the maximum value.

We have, therefore, this result. If the current in a linear conductor varies in strength and direction in a manner which geometrically would be represented by the ordinate of a simple

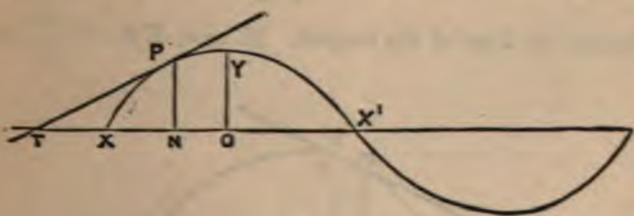

FIG. 15.

sine curve, the true mean value of the current strength is $\frac{2}{\pi}$, or ·637 of its maximum value, and the mean value of the square of the current strength taken at equal and very small intervals is half the value of the square of the maximum value.

Since $\frac{2}{\pi} = {\cdot}637$ and $\frac{1}{\sqrt{2}} = {\cdot}707$, and since the difference $= {\cdot}07$, the true mean current is less than the square root of the mean of the squares at each instant by an amount which is very nearly 10 per cent. of the latter.

§ 7. **Derived Sine Curve.**—The next geometrical proposition to which we pass is one of some importance.

Let the curve in Fig. 15 represent the complete period of a simple periodic curve of which the equation is $y = A \sin \frac{\pi}{l} x$. Let P be any point on the curve, then $PN = y$, $OY = A$, $XX' = l$. At P draw a tangent PT to the curve, and let it meet the datum line at T.

We shall call the trigonometrical tangent of the angle PTN, the *slope* of the tangent at the point P, hence $\frac{PN}{TN}$ = the slope.

If two points, PP′ (Fig. 16), are taken on the curve very near together, and a secant, P′PT, drawn through them, this secant will become a tangent when the points P P′ move up into contact. The ratio of $\frac{P'M}{PM}$ will then, in the limit, become the slope of the tangent. If, now, $XN = x - \frac{\delta x}{2}$, and

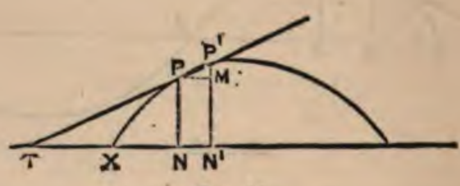

FIG. 16.

$XN' = x + \frac{\delta x}{2}$, and $PM = NN'$ is δx, we have the equations

$$PN = A \sin \frac{\pi}{l}\left(x - \frac{\delta x}{2}\right)$$

and
$$P'N' = A \sin \frac{\pi}{l}\left(x + \frac{\delta x}{2}\right)$$

hence
$$\frac{P'M}{PM} = \frac{A}{\delta x}\left[\sin \frac{\pi}{l}\left(x + \frac{\delta x}{2}\right) - \sin \frac{\pi}{l}\left(x - \frac{\delta x}{2}\right)\right]$$

The quantity in the bracket is identically the same as

$$2 \cos \frac{\pi}{l} x \sin \frac{\pi}{l} \frac{\delta x}{2},$$

and hence

$$\frac{P'M}{PM} = \frac{A}{\delta x}\,\frac{\pi}{l}\frac{\delta x}{2}\,2\cos\frac{\pi}{l}x\left[\frac{\sin\frac{\pi}{l}\frac{\delta x}{2}}{\frac{\pi}{l}\frac{\delta x}{2}}\right]$$

$$= \frac{A\pi}{l}\sin\left(\frac{\pi}{l}\left(\frac{l}{2}-x\right)\right)\left[\frac{\sin\frac{\pi}{l}\frac{\delta x}{2}}{\frac{\pi}{l}\frac{\delta x}{2}}\right]$$

When δx is made infinitely small the quantity in the square brackets is unity, and we have

$$\text{slope} = \frac{A\pi}{l}\sin\left(\frac{\pi}{l}\left(\frac{l}{2}-x\right)\right).$$

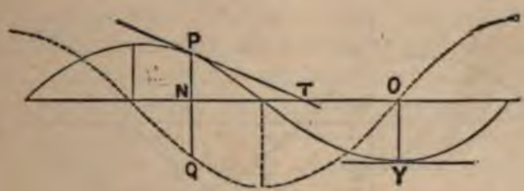

FIG. 17.

If we plot a curve whose ordinates at any point are the slope of the primal curve at the corresponding points, the above equation shows us three things—first, that it is a sine curve or simple periodic curve of the same type as the curve from which it is derived; second, that its maximum value is $\frac{\pi}{l}$ times the maximum value of the original; and third, that its zero ordinate corresponds to the maximum one of the original, and *vice versâ*.

In Fig. 17 the firm line curve is a curve of sines

$$y = A\sin\frac{\pi}{l}x,$$

the dotted line is a curve of sines, whose ordinate Q N at any point represents the *slope* of the tangent at P on the original curve. Accordingly at Y, where the original curve is at its maximum, and the slope of its tangent is zero, the derived curve cuts the datum line, or has its phase shifted 90deg. backwards relatively to the original curve. In the language of the differential calculus, the firm line curve is the plotting of $y = A \sin \frac{\pi}{l} x$, and the dotted curve is the plotting of $\frac{dy}{dx}$ as ordinates for the same abscissæ. We may regard it from another point of view. Let the simple sine curve be supposed to be generated or marked out by a tracing point, P, which moves to and fro along a line P N P′ with a simple harmonic motion, whilst the point N moves uniformly along a straight line X X′. (*See* Fig. 18.)

Draw as before the dotted curve whose ordinate Q N at any point represents the slope of the firm curve at the corresponding point P. Then the magnitude of N Q will represent the

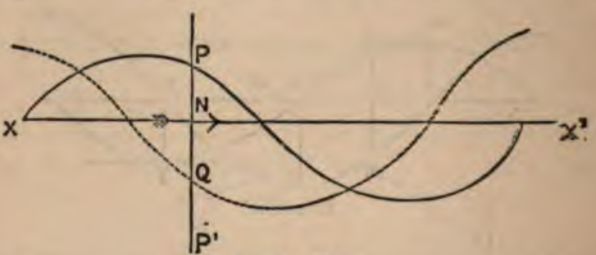

FIG. 18.

ing point P. Then the magnitude of N Q will represent the *rate* at which the ordinate P N is increasing or decreasing. For, in this case, distances such as X N, measured along the mean line, are proportional to time, and hence N makes a small movement forward in a small time $d\,t$; there is a corresponding decrease in the ordinate P N, which we may denote by $d\,y$, and accordingly $\frac{dy}{dt}$ represents the rate of decrease of P N. If the small forward movement of N causes N to advance through a space $d\,x$, $d\,x$ is proportional to $d\,t$, as the motion is uniform, and accordingly $\frac{dy}{dx}$ is proportional to $\frac{dy}{dt}$, hence $\frac{dy}{dt}$ is at any instant graphically represented by the slope of the tangent at P—that is, by the ordinate Q N. The dotted curve represents, therefore, the *rate of change* of the ordinates of the firm curve at that same instant. We shall call the dotted curve the *derived curve.*

If the original curve represents a simple periodic current flowing in a conductor, the derived curve will represent the *rate of change of that flow* at each instant. The derived curve is a similar curve, but shifted backwards by one quarter wave length, so that its crests coincide with the hollows of the original curve, and *vice versâ.*

§ 8. **Current Growth in Inductive Circuits.**—We proceed to examine the principles underlying the simplest case of periodic current flow in a linear conductor formed of non-magnetic material and not having any iron around it or near it. Let there be any conductor, such as a copper wire, of which the cross section is very small, and let this be formed into a coil or helix of many turns and subjected to a simple periodic electromotive force, called hereafter the *impressed electromotive force,* the periodicity being not greater than about 200–300 per second. It is required to find the current at any instant in the wire. The minds of many electrical students who have been educated too much or too exclusively by contact with continuous currents are apt to imagine that the simple and familiar law of Ohm is all that is required to meet this case. The supposition is, however, wrong, and the deduction would be found to be disappointed in practice. Every circuit possesses, as we have seen, two qualities, which determine the current when it is subjected to a simple periodic electromotive force of given periodicity. These are its *inductance* and its *resistance.* The resistance of a conductor is a quality in virtue of which the passage of a current through it is accompanied by the dissipation of energy—that is, by the irreversible transformation of electric energy into heat. The inductance of a circuit is a quality in virtue of which the passage of an electric current through it is accompanied by the absorption of energy in the form of a magnetic field. When electric energy is spent on a conductor (assuming no performance of chemical or external mechanical work) part fritters away into heat by an irreversible process, and part is associated with the circuit in a recoverable form, and is taken up in the establishment of the energy of the magnetic field, which starts into existence round the conductor. This portion of the energy, however, dissipates itself as soon as the impressed electromotive force is withdrawn. A mechanical operation analogous to that of starting a current in a wire may be found in the process of starting from rest, or increasing the speed of, a heavy fly-wheel which runs in bearings with friction. On applying a twisting force to the axle of the wheel we get up its speed. To maintain the speed force has to be continually applied, and the work so done against friction is frittered away irreversibly into heat in the bearings. The friction is analogous to the electrical resistance; it may be called the frictional resistance.

When the speed of the wheel is constant there is, however, associated with the wheel a certain quantity of energy in a kinetic form measured by $\frac{1}{2}\,I\,\omega^2$, where I is the moment of inertia, and ω the angular velocity of the wheel. As soon as the maintaining force is withdrawn this accumulated energy dissipates itself in heat by friction, or is utilised in some other way. During the time that the speed of the wheel is being increased, force must be applied to it for two purposes. First to increase the angular momentum, and secondly to overcome the friction at the bearings. Suppose that instead of revolving on bearings with friction, the fly-wheel revolves in a more or less viscous fluid, and that the bearings are truly frictionless. In such case the frictional resistance to motion would be fluid resistance, and would for low speeds be approximately proportional to the angular velocity. If I is the moment of inertia and ω the angular velocity of the wheel at any instant, then it is shown in treatises on dynamics that the product of the moment of inertia and the rate of change of the angular velocity at the instant, or $I\frac{d\omega}{dt}$, is the numerical measure of the couple or twisting force acting on the wheel to increase its angular velocity, friction being neglected. If we call the constant frictional coefficient B, so that $B\,\omega$ is at any instant the measure of the force necessary to maintain the motion against friction, the total torsional or twisting force acting on the wheel to maintain its angular velocity against the force of friction, and to increase it against the force of inertia, is

$$F = B\omega + I\frac{d\omega}{dt}.$$

A precisely similar equation may be found connecting the electromotive force, current electrical resistance, and inductance in the case of current starting in a wire. The above equation gives us a value for the instantaneous angular velocity, or enables us to find the angular velocity after any time when F, B, and I are given. When a current of strength i is flowing steadily in a linear conductor, such as the wire under consideration, the energy associated with it in the form of a magnetic field is, as we have seen in Chap. II., measured by the quantity $\frac{1}{2}Li^2$, where L is the quantity called the inductance of the circuit. Since this quantity L bears to electro-magnetic energy a relation similar to that which the moment of inertia of a wheel does to the energy of its rotation, it might be called the coefficient of electro-magnetic inertia; but as this would be a cumbersome name, it has been called the inductance, or more frequently the self-induction of the circuit. The numerical product of the moment of inertia and the angular velocity of the wheel is called the angular momentum, and analogously the product of the inductance of a circuit and the current flowing at that instant through it is called the electro-magnetic momentum. The rate at which the angular momentum of a wheel is increasing or diminishing at any instant is a measure of the rotational force, or the couple acting on it at that instant. So also the rate of change of the electro-magnetic momentum of a circuit is the measure of the electromotive force acting on it as far as mere change of current strength is concerned, and omitting, for the present, that part of the electromotive force required to overcome the true resistance. We have, then, the following parallel between a fly-wheel, with moment of inertia I, revolving frictionlessly, and having an angular velocity ω at any instant, and an electric circuit of inductance L, having a current of strength i flowing in it at any instant:—

Angular kinetic energy of the wheel, or energy of rotation	$= \frac{1}{2} I \omega^2$
Electro-magnetic energy of the circuit	$= \frac{1}{2} L i^2$
Angular momentum of wheel	$= I \omega$
Electro-magnetic momentum of the circuit	$= L i$
Rate of change of angular momentum of wheel = couple or torsional force causing rotation	$= I \frac{d \omega}{d t}$
Rate of change of electro-magnetic momentum = electromotive force employed in changing current strength	$= L \frac{d i}{d t}$

The symbol (=) must in the above be understood as equivalent to the phrase "is measured by."

In the electric circuit, over and above the electromotive force which is, so to speak, used up in changing electro-magnetic momentum, there is an amount required to overcome the frictional resistance of the wire, and which, as it is defined and measured by Ohm's law $E = R i$, may be called the Ohmic resistance. Hence, at any instant, if E is the impressed electromotive force acting on the circuit and causing a current i, we may divide E into two parts, one part equal to Ri by Ohm's law, which is sometimes called the effective electromotive force, and another part equal to $L \frac{d i}{d t}$, which is the part operating to change the strength of the current at that instant, producing a small change, $d i$, in the current strength i in a time $d t$.

Hence $$E = R i + L \frac{d i}{d t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the fundamental equation for varying or periodic currents, when the periodicity is not so rapid as to affect the uniform distribution of the current over the cross section of the wire, and when the electrostatic capacity may be neglected. The part $L \frac{d i}{d t}$ is often called the counter-electromotive force of self-induction, and the above equation might be read in words—

$$\text{Impressed E.M.F.} = \text{Effective E.M.F.} + \text{Counter E.M.F. of self-induction.}$$

We might otherwise arrive at this fundamental equation thus: The total rate of expenditure of energy in the circuit is at any instant measured by the product of the current at that instant existing in the wire and the difference of potential between its ends. The energy expended in the circuit is at any instant being partly dissipated at a rate equal to $\mathrm{R}\,i^2$, R being the Ohmic resistance, and i the current, and partly being stored up in the field at a *rate* equal to the rate of change of the quantity $\frac{1}{2}\mathrm{L}\,i^2$. Hence we have:—

$$\text{Rate of supply of energy} = \text{Rate of dissipation of energy as heat} + \text{Rate of absorption or storage of energy in the magnetic field.}$$

and this in symbols is—

$$\mathrm{E}\,i = \mathrm{R}\,i^2 + \frac{d}{dt}\left(\frac{1}{2}\mathrm{L}\,i^2\right)$$

or

$$\mathrm{E} = \mathrm{R}\,i + \mathrm{L}\frac{d\,i}{d\,t}, \quad \ldots \ldots (1)$$

which is our fundamental equation.

At this stage we must particularly caution the student to note one thing. The quantity L, which we have called the inductance of the circuit, is a constant and definite numerical quantity for any given form of circuit, as long as this circuit consists of non-magnetic material and is immersed in a non-magnetic medium. If, however, the circuit embraces or is embraced by iron, as in the case of an electro-magnet, or is immersed in a medium which is not diamagnetic but magnetic like iron, then, as we have seen, the inductance varies with the strength of the current flowing in the circuit, and it can no longer be considered as a constant quantity. In this chapter we suppose ourselves dealing only with circuits of constant inductance, and in which the value of L is fixed by the form of the circuit alone.

§ 9. **Equation for Establishment of a Steady Current.**—We return to our discussion of equation 1 (§ 8). When a current is flowing in a conductor, we may picture it as surrounded by its lines of electro-magnetic force *properly* mapped out. That is, so that the number of the lines of force passing perpendicularly through a small unit of area taken at any point in the field is equal to the numerical value of the mean strength of the magnetic field over that area. If the circuit has the form of a loop (Fig. 19) lying on a horizontal plane, with the current circulating round it in the opposite direction as the hands of a watch rotate, then the lines of force must be considered as springing out from the upper surface, and turning outwards and over the conductor, so as to re-enter the loop from the under surface. The closed circuit is, therefore, linked with a certain number of lines of force, which, if the circuit is composed of non-magnetic material, are proportional in number to the strength of the current at that instant. Any

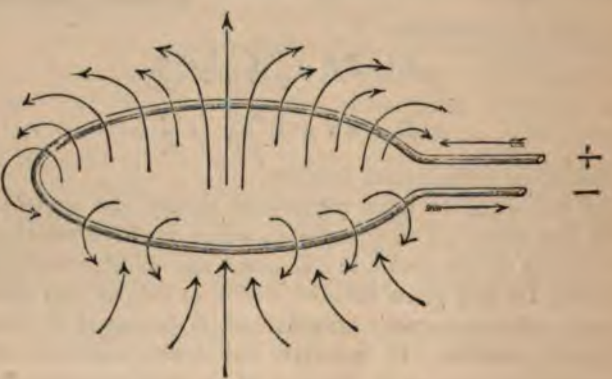

FIG. 19.

increase in strength of the current causes more lines of force to grow out from the circuit, and packs the loop fuller of lines of force. By Faraday's law, any increase of the number of lines of force traversing or linked with a circuit creates an induced electromotive force numerically equal to the rate of increase of that number at that instant. Hence, if 100 million lines of force—C.-G.-S. measure—are put or inserted at a uniform rate in one second into a circuit, it will create an induced E.M.F. of one volt in it. If lines of force are thrust into a circuit the direction of the current induced is counter clock-wise, as seen from that side of the circuit at which they are thrust in (*see* Fig. 20).

Applying this to the case before us, it is easily seen that any increase of current strength in the circuit in Fig. 19 crowds the space with more lines of force, and therefore creates in it an electromotive force of self-induction opposed to the impressed electromotive force which is acting to increase the current, and as long as the current is increasing, this counter E.M.F. is at each instant proportional to the rate of growth of the current strength.

We can cast our current equation—

$$E = Ri + L\frac{di}{dt}$$

into another form, thus:—

$$\frac{E}{R} - i = \frac{L}{R}\frac{di}{dt},$$

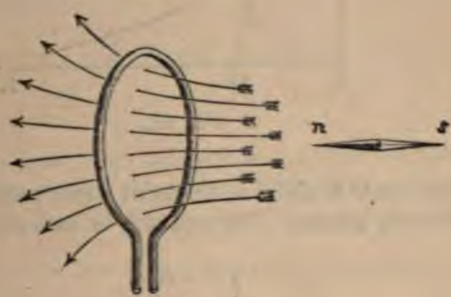

Fig. 20.

Lines of force being crowded into a circuit, inducing a counter clockwise E.M.F., as seen from the side at which they are put in.

where $\frac{E}{R}$ is the maximum value which the current can attain; let us call this I. The quantity $\frac{L}{R}$, or the ratio of the inductance to the resistance of the circuit, is called the *time-constant* of the circuit; let this quantity be denoted by T. We then have

$$I - i = T\frac{di}{dt},$$

which, in words, is a statement that if a steady E.M.F. is made to act on any circuit whose time constant is T, the amount by which at any instant the current falls short of its full value is equal to its rate of growth at that instant, multiplied by the time-constant.

§ **10. Logarithmic Curves.**—A curve such that the rate of growth or shrinkage of the ordinate is proportional to the ordinate itself is called a logarithmic curve.

Let a curve (Fig. 21) be described by the extremity P of an ordinate, P M, which moves uniformly along O X, parallel to

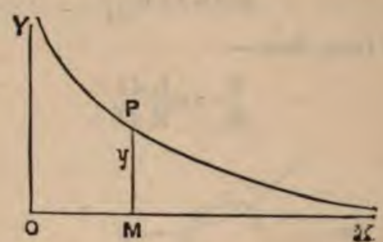

Fig. 21.

itself, and let P M *shrink* in height at a rate proportional to its height at any instant. The equation to such a curve is

$$y = -A\frac{dy}{dt},$$

and since e^{-Bx} (where e = the base of Naperian logarithms = 2·71828) is a function which fulfils this condition of having a differential coefficient proportional to itself, we can write the solution of the above

$$y = e^{-\frac{t}{A}},$$

for it is at once seen that by differentiating the equation

$$y = e^{-\frac{t}{A}}$$

we obtain
$$\frac{dy}{dt} = -\frac{e^{-\frac{t}{A}}}{A}$$

and therefore
$$y = -A\frac{dy}{dt}.$$

Returning to our equation for the current we can write a an equivalent for the equation

$$I - i = T\frac{di}{dt}$$

the equation
$$I - i = -T\frac{d(I-i)}{dt},$$

or
$$\frac{dt}{T} = -\frac{d(I-i)}{I-i}.$$

Integrating this we have as a solution

$$-\frac{t}{T} = .\log(I-i) + \text{const.}$$

The constant has to be determined by the condition that, when $t=0$, $i=0$, which gives const. $= -\log I$. Hence the complete solution is

$$-\frac{t}{T} = \log(I-i) - \log I,$$

or
$$I - i = I e^{-\frac{t}{T}}.$$

This last equation expresses the fact that the amount by which the current falls short of its full value I, at any time t, after applying the E.M.F., is a fraction of its full value, equal to $e^{-\frac{t}{T}}$. When $t=0$, or at the instant of closing circuit, $I-i=I$, or the current $i=0$; when $t=T$, $I-i=\frac{I}{e}$, or the deficit from full current is equal to $\frac{1}{2\cdot718} \times$ the maximum current. Hence we may define the *time-constant* of a circuit as the time reckoned from the instant of closing the circuit in which the current rises up to a value equal to $\frac{e-1}{e}$ of its full value, or to about ·632 of its maximum value. Approximately we may define the time-constant as the time from closing the circuit in which the current rises up to ⅔rds of its maximum value $\frac{E}{R}$.

The rise of current strength in a wire of inductance L and resistance R when a steady external electromotive force, E, is applied to the circuit can be represented by a current curve, as shown in Fig. 22.

Let O X (Fig. 22) be a time line on which we mark off time as lengths reckoned from O. Let lines drawn vertically to this represent the current strength at any instant in a circuit of time-constant T, inductance L, and resistance R, and let $OY = I = \frac{E}{R}$ represent the maximum current which is finally found in the circuit. On applying the electromotive force E to the circuit, the current strength grows up in the wire as graphically represented by the curve, the law of growth being that the rate of growth at any instant, multiplied by the time-

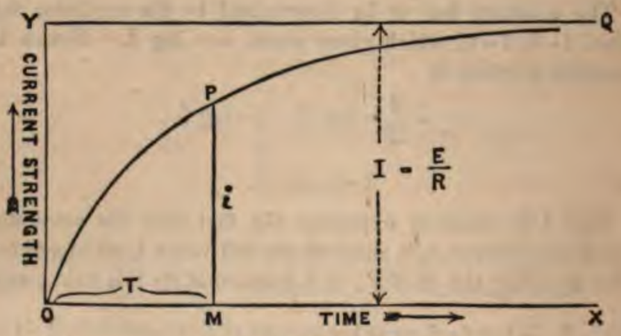

FIG. 22.

constant, is equal to the difference between the actual current at that instant and the maximum current strength finally attained, or symbolically

$$I - i = T\frac{di}{dt},$$

the solution of the above differential equation being

$$I - i = I\epsilon^{-\frac{t}{T}}$$

or

$$i = I\left(1 - \epsilon^{-\frac{t}{T}}\right) \quad . \quad . \quad . \quad (2)$$

This last equation gives us the value of the current strength at any time t seconds after closing the circuit in terms of the time-constant and the maximum current I which is finally attained.

The maximum current I would be produced at once in the circuit if its inductance were zero, so that we may finally formulate the law of growth of current in a circuit of constant inductance L, resistance R, and no sensible capacity, by saying that *the current strength at any instant, added to the rate of growth of the current strength at that instant multiplied by the time-constant, is equal to the current which would exist in the circuit if inductance were zero.*

§ **11. Instantaneous Value of a Simple Periodic Current.**— The application of this principle to the case of simple periodic currents will lead to another proposition. Let there be a circuit which has an inductance L and resistance R, and let a simple periodic electromotive force act upon it. Let the maximum value of this E.M.F. be E, and let p stand for $2\pi n$, where n is the frequency of the oscillations, or $\frac{1}{n}$ is the duration of one single complete period; p is a quantity of the nature of an angular velocity, and may be called the *pulsation*. Then if t is the time which has elapsed from the commencement of the wave of E.M.F., and e is the actual value of the E.M.F. at that instant,

$$e = \text{E} \sin p t.$$

In this case the impressed electromotive force varies from instant to instant, passing from zero to a maximum, E, then to zero again, and then to a negative maximum $-\text{E}$. Accordingly our fundamental equation for the current strength at any instant is expressed thus,

$$\frac{d(\text{L}i)}{dt} + \text{R}i = e = \text{E} \sin p t \quad . \quad . \quad . \qquad (3)$$

For, the total rate of expenditure of work on the circuit at any instant when the current has a value i is $e i$, and this must be equal to the rate at which electrical work is being dissipated as heat, or to $\text{R}i^2$ by Joule's law, and to the rate at which work is being stored up in the magnetic field, which is

$$\frac{d}{dt}(\tfrac{1}{2}\text{L}i^2);$$

hence,
$$\frac{d}{dt}(\tfrac{1}{2}\,\mathrm{L}\,i^2)+\mathrm{R}\,i^2=e\,i,$$

or,
$$\mathrm{L}\frac{di}{dt}+\mathrm{R}\,i=\mathrm{E}\sin p\,t.$$

In order to solve this differential equation and obtain the value of the current i in the circuit at any instant under the periodic electromotive force, we may adopt a well-known algebraic device, and substitute for the value of $\sin p\,t$ its equivalent in exponential terms. It is shown in treatises on trigonometry that

$$\sin\theta=\frac{e^{k\theta}-e^{-k\theta}}{2k},$$

where $k=\sqrt{-1}$, and e is the number 2·71828, which is the base of the Naperian logarithms.

Also
$$\cos\theta=\frac{e^{k\theta}+e^{-k\theta}}{2};$$

hence $\cos\theta+k\sin\theta=e^{k\theta}$.

These are called the exponential values of the sine and cosine.

Taking the equation

$$\mathrm{L}\frac{di}{dt}+\mathrm{R}\,i=\mathrm{E}\sin p\,t,$$

we divide both sides by L, and writing T as before for the *time-constant* $\frac{\mathrm{L}}{\mathrm{R}}$, we get

$$\frac{di}{dt}+\frac{i}{\mathrm{T}}=\frac{\mathrm{E}}{\mathrm{RT}}\sin p\,t.$$

Multiply both sides by $e^{\frac{t}{\mathrm{T}}}$ (e being here the exponential base, not impressed E.M.F.), and we have

$$\frac{di}{dt}\cdot e^{\frac{t}{\mathrm{T}}}+\frac{i}{\mathrm{T}}\,e^{\frac{t}{\mathrm{T}}}=\frac{\mathrm{E}}{\mathrm{RT}}\,e^{\frac{t}{\mathrm{T}}}\sin p\,t.$$

The left hand side of this equation is the complete differential of $i\,e^{\frac{t}{T}}$, and may be written $\frac{d}{dt}\left(ie^{\frac{t}{T}}\right)$, and on substituting the exponential value for sin $p\,t$ and putting k for $\sqrt{-1}$, we have

$$\frac{d}{dt}\left(i\,e^{\frac{t}{T}}\right)=\frac{E}{2\,k\,R\,T}\left\{e^{\left(\frac{t}{T}+kpt\right)}-e^{\left(\frac{t}{T}-kpt\right)}\right\}. \quad (4)$$

The right-hand side of this last equation is the differential with respect to t of

$$\frac{E}{2\,k\,R\,T}\left\{\frac{e^{\left(\frac{1+kpT}{T}\right)t}}{\frac{1+k\,p\,T}{T}}-\frac{e^{\left(\frac{1-kpT}{T}\right)t}}{\frac{1-k\,p\,T}{T}}\right\},$$

and this last becomes by simplification

$$\frac{E}{2\,k\,R}\,e^{\frac{t}{T}}\left\{\frac{e^{kpt}}{1+k\,p\,T}-\frac{e^{-kpt}}{1-k\,p\,T}\right\};$$

hence equating both sides of equation (4), when integrated we have

$$i=\frac{E}{2\,R\,k}\left\{\frac{e^{kpt}}{1+k\,p\,T}-\frac{e^{-kpt}}{1-k\,p\,T}\right\}.$$

Substituting back into sine and cosine terms, and recollecting that

$$e^{kpt}=\cos p\,t+k\sin p\,t,$$

and

$$e^{-kpt}=\cos p\,t-k\sin p\,t,$$

we get finally

$$i=\frac{E}{R}\left\{\frac{\sin p\,t-p\,T\cos p\,t}{1+p^2\,T^2}\right\}.$$

If $p=0$—that is, if the pulse is infinitely long, or the electromotive force continuous—then we have simply $i=\frac{E}{R}$, which is Ohm's law; but when p is not zero, and we have periodic variation of E.M.F., we see that the current value is not obtained by dividing E by R, but by multiplying this quotient by a factor.

We can, however, put the above equation in a more intelligible form. Replace T by $\frac{L}{R}$, and let θ be an angle whose

tangent is
$$\frac{Lp}{R};$$

hence
$$\tan\theta = \frac{Lp}{R} = pT.$$

It follows by an easy transformation that

$$\cos\theta = \frac{R}{\sqrt{R^2+p^2L^2}}, \qquad \text{and } \sin\theta = \frac{Lp}{\sqrt{R^2+p^2L^2}}.$$

We have then for the value of the current

$$i = \frac{E}{R}\left\{\frac{\sin pt - pT\cos pt}{1+p^2T^2}\right\};$$

or, by substitution,

$$i = \frac{E}{\sqrt{R^2+p^2L^2}}\left\{\sin pt\cos\theta - \sin\theta\cos pt\right\}.$$

$$\therefore i = \frac{E}{\sqrt{R^2+p^2L^2}}\sin(pt-\theta).$$

This is the final solution of the equation

$$L\frac{di}{dt} + Ri = E\sin pt = e \quad . \quad . \quad . \quad . \quad (4)$$

and it shows us three things. First, that the phase of the current i is retarded behind that of the impressed electromotive force by an angle θ—such that $\tan\theta = pT = \frac{Lp}{R}$; and second, that the maximum value of the current is obtained by dividing the maximum value of the electromotive force by a quantity equal to $\sqrt{R^2+p^2L^2}$; and, third, that the current curve is a simple periodic curve. The quantity $\sqrt{R^2+p^2L^2}$ is called the *impedance* of the circuit.

We have seen from the explanations on previous pages that the average or true mean value of a simple periodic quantity is equal to its maximum value multiplied by $\frac{2}{\pi}$. Hence, if we write Im for impedance, we can put the equation, giving the value of the current produced by a simple periodic impressed electromotive force of maximum value E, operating on a circuit of resistance R, inductance L, with a pulsation p, in the form

$$i = \frac{E}{Im} \sin(pt - \theta),$$

and in words

$$\begin{matrix} average \\ current \\ strength \end{matrix} = \frac{\text{average impressed electromotive force}}{\text{impedance}}$$

or

$$\begin{matrix} maximum \\ current \\ strength \end{matrix} = \frac{\text{maximum impressed electromotive force}}{\text{impedance}}.$$

We see, then, that in the case of simple periodic electromotive force the quantity called the impedance appears to be related to the impressed E.M.F., just as does the resistance to the steady E.M.F. in the case of continuous currents, and the above may be called the equivalent of Ohm's law for alternate currents. Compare as below

For steady or continuous currents $\begin{matrix} current \\ strength \end{matrix} = \frac{\text{electromotive force}}{\text{resistance}}$ (Ohm's law).

For simple periodic or alternate currents $\begin{matrix} average \\ current \\ strength \end{matrix} = \frac{\text{average impressed electromotive force}}{\text{impedance}}.$

Instead of *average*, we may write in the above *maximum*, or *square root of mean square* of the current.

Impedance is a quantity which is measured, like resistance, in ohms, and has for that reason been sometimes called the *virtual resistance.*

§ **12. Geometrical Illustrations.**—The current equation, expressing the current strength in terms of the impressed electromotive force, the resistance, inductance, and phase angles, which holds good when a circuit of constant inductance and no sensible capacity is subjected to moderately great pulsations of electromotive force, has been in the previous pages arrived at algebraically from first principles. It is, however, possible to elucidate its meaning by geometrical methods. Let a circular disc (Fig. 23) be pivoted at the centre O, and at any point P on the circumference let a plummet line be attached. In front of the circle is a fixed horizontal line $X X^1$. Let the disc move round counter-clockwise at a uniform rate, the time of one revolution being T. As the disc goes round, the length of plummet line P M above $X X^1$ fluctuates. Since $PM = OP \sin POM$, it follows that if the magnitude of P M be taken at small equal intervals of time during one revolution, and such heights be plotted off as off-sets at equal distances above and below a datum line, the extremities of these ordinates will lie on a simple periodic or sine curve. In

Fig. 23.

other words, P M grows and shrinks in height in accordance with a simple periodic law. We can, therefore, represent any quantity which fluctuates in magnitude according to a simple sine law of growth by representing it as the projection of a point on the circumference of a circle revolving uniformly, taken on a horizontal or vertical fixed line drawn through the centre. Hence, if O P represent the maximum value of an electromotive force fluctuating periodically, P M will represent its various magnitudes during the complete period. The magnitude of P M at any instant is known when we know O P, which is called the *amplitude* or maximum value, and P O M the *phase angle* of the motion.

§ **13. Graphic Representation of Periodic Currents.**—On such a diagram let a radius be drawn to any scale representing by its vertical projection the periodic fluctuation of an impressed electromotive force varying according to a simple sine law, and acting on a circuit of given inductance and resistance with a fixed periodicity; the problem is to draw on the same diagram another radius of which the vertical projection shall represent the actual current strength in the circuit at the corresponding instant. The impressed electromotive force at any instant balances, or is equal to, the sum of two others, viz., the effective electromotive force driving the current, which is equal to the product of the Ohmic resistance of the circuit and the current at that instant in it, and the inductive or counter-electromotive force, which is equal to the rate of variation of the flux of force or number of lines of force traversing the circuit. The phases, or times of maximum, of these two components are not identical. They differ by 90°, since the effective electromotive force has the same phase as the actual current, and the inductive electromotive force, depending on the *rate of variation* of the current, comes to a maximum at the instant when the current is zero or is changing sign.

By the proposition in § 7, these two periodic quantities can therefore be represented by sine curves, one of which is shifted backward relatively to the other, so that the crest of the wave of one coincides with the hollow of the wave of the other. We shall first proceed to show that the sum of two simple periodic motions of the same periodic time, but different phases and amplitudes, will, when added together, produce a simple periodic motion of the same periodic time.

Let a parallelogram of cardboard, O A B C (Fig. 24), be cut out and pivoted by a pin at the angle O, so as to turn freely clockhand-wise. Let a vertical line, O Y, be drawn through O, and in any position let the sides O A, O C, A B be projected on to O Y. The projection of lines equal and equally inclined are equal; hence, since A B is equal and parallel to O C, the projection of A B—viz., $a\,b$—is equal to that of O C—viz., O c. But $O\,b = O\,a + a\,b$ always for any position of the card; hence $O\,b = O\,a + O\,c$. The projection of the diagonal is therefore equal to the sum of the projections of the adjacent sides.

As the card moves uniformly round, the magnitudes of the projections fluctuate at each instant, according to a simple periodic law. Hence the sum of the simple periodic motions of which O A, O C are the amplitudes, and which have a fixed difference of phase represented by the angle A O C, is the simple periodic motion represented by O B in amplitude and relative phase. If, then, a point be subjected to two simultaneous simple periodic motions of given amplitudes, and of which the phases differ by 90°, the actual motion will be represented, as to amplitude and phase, by the diagonal of the parallelogram of which these two form the adjacent

Fig. 24.

sides. Returning in thought to electric motion, consider the motion of a *particle of electricity* (if we may be allowed the expression) in the wire subjected to two simultaneous simple periodic motions of unequal amplitude and fixed difference of phase equal to 90°. The displacement at any instant due to the two together is equal to the sum of each separately. If the individual motions are represented by the vertical projections of two lines, O A, O B, fixed like hands of a toy clock at right angles (Fig. 25), the resultant motion is that indicated by the projection of the diagonal O C on the same vertical. We have seen (in § 7) that, if the variation of a quantity is represented by a simple sine curve, the variation of its *rate of change* is represented by a sine curve of different amplitude shifted backwards by 90° of phase, or by a quarter of a wave length. It follows from this proposition that if we add together at every instant the motions or the ordinates representing them on a diagram of two simple periodic motions, one of which is the curve representing the rate of change of the ordinate of the other, we shall get a new sine curve, of which the maximum value falls between that of the other two, and of which the amplitude is different, but wave length or periodic time the same. In Fig. 26 the thick white line sine curve represents one wave of a simple periodic motion. The fine white continuous line is a sine curve of equal wave

FIG. 25.

length, of which the ordinate P M at any point represents or is proportional to the rate of change of the ordinate Q M of the thick curve at the same instant. Adding together the ordinates of the thick and thin curves, we get a new dotted line sine curve, of which the ordinate R M is equal to Q M + rate of change of Q M. If we substitute for the sine curve diagram a clockhand diagram (Fig. 25), then the projection of O B, viz., O *b*, corresponds to the ordinate Q M of the black curve; that of O A, viz., O *a*, corresponds to P M, the ordinate of the thin curve; and that of O C the diagonal of the rectangle, O A, O B, corresponds to R M, and is the resultant of the motion O B, and the rate of change of that motion, viz., O A. If, then, O B represents the amplitude or maximum value of the actual periodic current in a circuit, a line, O A, drawn at right angles to O B, will represent to a suitable scale the rate of change of that current.

We are, then, led to this converse proposition, that we can resolve any simple periodic curve into a pair of component periodic curves of equal periodic time, but of which the maximum value happens for one before, and for one after, that of the original. It is not difficult to see that the geometrical construction necessary is as follows. Let a unit of length be taken to represent a volt, an ohm, an ampere, and a unit of self induction equal to an earth quadrant or 10^9 centimetres.* Describe a circle whose radius is numerically equal to

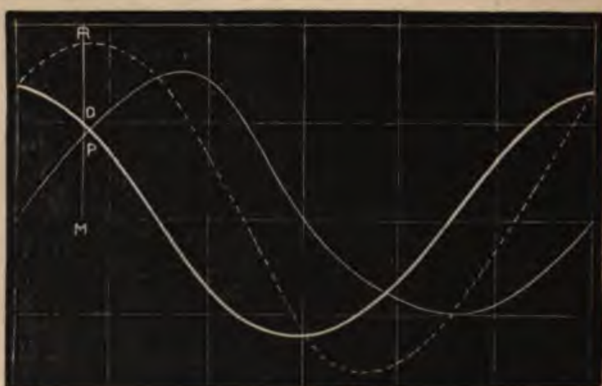

FIG. 26.

the maximum value of the impressed electromotive force (Fig. 27). Draw any radius O P to represent the amplitude E of this impressed E.M.F. If, then, the angle P O X is represented by *pt*, the value of the vertical projection of O P, viz., P M, is E sin *pt*, and is therefore the instantaneous impressed E.M.F. corresponding to the phase *pt*. On O P describe a semicircle, and set off O Q, making an angle P O Q $= \theta$ with O P, such that the tangent of P O Q is equal to the

* The unit of self-induction, whose value is an earth quadrant or 10^9 centimetres, has not yet been formally christened. Profs. Ayrton and Perry have called it a *secohm*, and Mr. Oliver Heaviside has proposed, I believe, to call it a *Mac* (after Maxwell). Not to introduce further confusion, we may tentatively adopt the former term.

ratio Lp to R. R is the ohmic resistance of the circuit, and Lp is equal to 2π times the quotient of the inductance of the circuit by the periodic time of the impressed E.M.F., or $p = 2\pi n$ as before. Join P Q and draw O W parallel to and equal to P Q. If I represents the maximum value of the current, and i its instantaneous value corresponding to the instant when its phase has certain value $pt - \theta$, we know that $i = \text{I} \sin (pt - \theta)$, and the rate of change of the current at that instant is then $\frac{di}{dt} = p\text{I} \cos (pt - \theta)$. The ratio of the maximum value of Ri to L $\frac{di}{dt}$ is therefore R I to LpI, or R to Lp; but Ri is the instantaneous effective electromo-

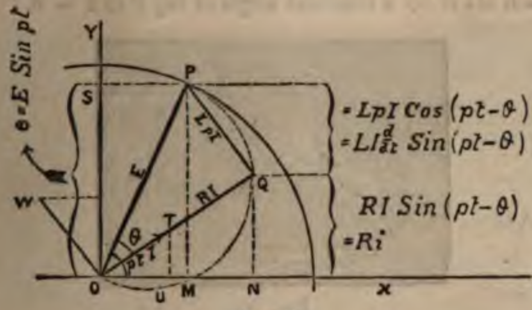

FIG. 27.

tive force driving the current, and L $\frac{di}{dt}$ is the corresponding counter electromotive force of self-induction at the same instant, and the sum of these two is the impressed electromotive force at that instant. Consider the two lines O Q and O W, the ratio of their magnitudes is R to Lp by construction, and the sum of their vertical projections is equal to the vertical projection of O P—that is, to the instantaneous impressed electromotive force. Hence these lines O Q and O W satisfy the conditions requisite for being the representation of the maximum effective and inductive electromotive forces respectively on the same scale and relative phase as that on which the line O P represents the maximum impressed electromotive force, and O Q therefore represents in magnitude the value of R I and O W that of Lp I, and the angular positions of these with respect to O P represent the relative phases of the effective, inductive, and impressed electromotive forces.

One R^{th} part of O Q is then the magnitude of the maximum current, and its phase of maximum value is behind that of O P by an angle θ. If O T = one R^{th} part of O Q, O T represents the maximum current, and T U, the vertical projection, represents the magnitude of the current in the circuit at the instant when the impressed electromotive force has the value P M.

The respectively varying current and impressed E.M.F. will be graphically represented by the variation of T U and P M as O P and O T move round the circle in equal times, separated from each other by a constant angle of lag P O T $= \theta$. It will

FIG. 28.

be seen that whilst O Q is equal in magnitude to the product of the true or ohmic resistance of the circuit and the maximum current that O W or P Q is equal to the product of Lp and the maximum current. Hence we may call Lp the *inductive resistance* of the circuit, because it is a quantity of like dimensions to the true or ohmic resistance. The angle of lag of the current is therefore an angle of which the tangent is the ratio of the inductive to the ohmic resistance.

If we draw a right angle triangle (Fig. 28), of which the base represents the ohmic resistance of the circuit and the perpendicular the inductive resistance, then the hypothenuse represents the quality $\sqrt{R^2 + L^2 p^2}$, which has been called the impedance. The angle of lag may therefore be defined thus,

$$\text{tangent of angle of lag} = \frac{\text{inductive resistance of circuit}}{\text{ohmic resistance of circuit}},$$

or

$$\text{cosine of angle of lag} = \frac{\text{ohmic resistance of circuit}}{\text{impedance of circuit}}.$$

In order to carry out the construction in Fig. 27 we require to know either tan θ or cos θ. The latter is most easily determined. The experimental determination of this quantity will be dealt with in due course.

§ 14. **Mean Value of the Power of a Periodic Current.**—Having now seen how the fluctuation of current strength is related to that of the impressed E.M.F. in a circuit under the conditions of a simple sine law of variation, we pass to the consideration of the measurement of the *power* or rate of

FIG. 29.

transformation of energy in the case of circuits traversed by simple periodic currents.

Let the thin line curve in Fig. 29 represent the sinusoidal impressed E.M.F. in an inductive circuit, and the thick white line the corresponding current. Then *at any instant* the rate at which energy is being expended on the circuit is equal to the product of the ordinates P M, Q M, which at any point M represent the E.M.F. and current respectively. The mean rate of expenditure of energy, or the *mean power* being taken up in the circuit, is then the mean of all such products taken at very numerous and very near intervals during one complete period. This is not by any means identical with the product of the mean current and mean electromotive force. To arrive at an expression for this mean power we must pave the way by a preliminary proposition on the mean product of two simple periodic quantities. An elegant geometrical proof of this has been given by Mr. Blakesley in his book ("Alternating Currents," page 10, *Electrician* series*). We shall, however, give here an algebraical proof of this proposition. Let there be two radii O P, O Q (Fig. 30), which revolve in equal periodic times round a common centre O, separated by a fixed angle, P O Q. At equal small intervals of time corresponding to equal angular motions let the projections O p, O q of these lines be taken on a vertical line through O. It is required to find the mean value of the product O p, O q during one complete period.

Denote by X the length of O P, and by Y the length of O Q, and let the angle P O Q be β, and P O p be α. β is the angle

FIG. 30.

of phase difference, and X and Y are the maximum values of the periodic quantities O p, O q, which are the vertical projections of O P, O Q.

Let O p be denoted by p, and O q by q.

Then $p = X \cos \alpha$,

and $q = Y \cos (\alpha + \beta)$;

and therefore $p q = X Y \cos \alpha \cos (\alpha + \beta)$.

Let the pair of radii O P, O Q be supposed to turn round one complete revolution, proceeding by n jumps, each jump or step increasing the angle α by a very small amount, $\delta \alpha$, and n being a very large number. At each stage let the value of $p q$

* "Alternating Currents." By T. H. Blakesley, M.A. (London: *The Electrician* Office, 1, Salisbury-court, Fleet-street, London, E.C.)

be measured as above, then the mean value of the product $p\,q$ is one nth part of the sum of all the n values so taken. Call this mean value of the product M. Then,

$$M = \frac{XY}{n}\left\{\begin{array}{r}\cos\beta + \cos\delta\alpha\cos(\delta\alpha+\beta) + \cos 2\,\delta\alpha\cos(2\,\delta\alpha+\beta) \\ \ldots\ldots + \cos\overline{n-1}\,\delta\alpha\cos(\overline{n-1}\,\delta\alpha+\beta)\end{array}\right\}.$$

By trigonometry we have

$$\cos(\overline{n-1}\,\delta\alpha)\cos(\overline{n-1}\,\delta\alpha+\beta) = \tfrac{1}{2}\cos(2\,\overline{n-1}\,\delta\alpha+\beta) + \tfrac{1}{2}\cos\beta,$$

since $\cos\overline{A+B} + \cos\overline{A-B} = 2\cos A\cos B.$

Accordingly every term, except the first in the cosine series for M, splits up into the sum of two others, one of which is always $\frac{1}{2}\cos\beta$. Rearranging the terms, we get for the value of M as follows :—

$$M = \frac{X\,Y}{n}\left[\tfrac{1}{2}\,n\cos\beta + \tfrac{1}{2}\Big(\cos\beta + \cos(2\,\delta\alpha+\beta) + \cos(4\,\delta\alpha+\beta) \ldots\ldots + \cos(2\,\overline{n-1}\,\delta\alpha+\beta)\Big)\right]$$

The cosine series in the inner bracket consists of a series of cosines of angles in arithmetic progression taken all round the circle. Hence, since the cosine of any angle is numerically equal to that of the cosine of its supplement, but of opposite sign, these cosine terms will cancel each other out pair and pair, when n becomes very great and $\delta\alpha$ very small, and $n\,\delta\alpha$ equal to 2π. For when

$$n\,\delta\alpha = 2\pi,\ \ 2\,\overline{n-1}\,\delta\alpha + \beta = 4\pi + \beta,\ \text{and}\ \cos(4\pi+\beta) = \cos\beta.$$

The first and last terms of the series are in this case identical, and for every term there will exist one of equal magnitude and opposite sign. The sum of the series of cosine terms in the inner bracket is accordingly zero.

The value of M reduces them to that of the first term, viz. :—

$$M = \frac{X\,Y}{2}\cos\beta.$$

The mean value of the product of two simple harmonic or periodic functions of equal period but different amplitude and phase, is equal to half the product of their maximum values, and the cosine of their difference of phase.

Returning to the consideration of the electrical problem, it is now clear that the mean value of the product of the current at any instant and the simultaneous value of the impressed electromotive force in an inductive circuit is obtained by multiplying together half the product of their maximum values, and the cosine of the angle of lag. If i be the current at any instant, and e the impressed E.M.F., I and E being their maximum values, then the mean value of $e\,i$ during a complete period is

$$\frac{\mathrm{E\,I}}{2}\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

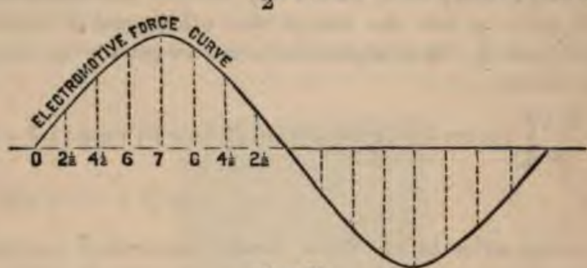

FIG. 31.

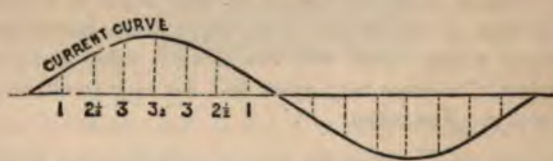

FIG. 32.

and this is a measure of the rate of expenditure of energy on that circuit, or the work expended per second. It is obvious, then, that if the lag is 90°, this mean product is zero, and that no work is done at all.

When θ has intermediate values between 0° and 90° the real rate of dissipation or transformation of energy in the circuit will be intermediate between $\frac{\mathrm{E\,I}}{2}$ and zero. In order to understand how this can be, and how it is that a circuit may be traversed by a current and yet take up no power, we must examine a little more closely the nature of the phenomena.

§ 15. **Power Curves.**—Let the thin periodic curve in Fig. 31 represent a sinusoidal variation of electromotive force acting on a circuit which we shall for the moment assume has no sensible inductance. Let the thick curve in Fig. 32 represent the corresponding current. The length of each ordinate of the thick curve is equal in magnitude to that of the corresponding ordinate of the thin curve divided

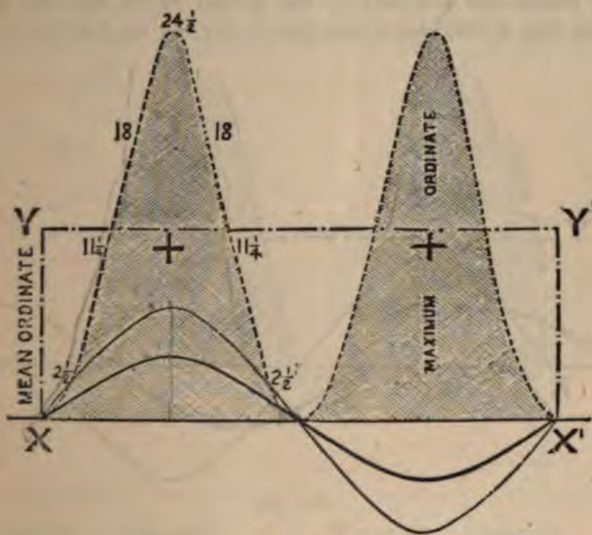

Fig. 33.

Ordinates.	Product.	Ordinates.	Product.
E × C =	P	7 × 3½ =	24½
0 × 0 =	0	6 × 3 =	18
2½ × 1 =	2½	4½ × 2½ =	11¼
4½ × 2½ =	11¼	2½ × 1 =	2½
6 × 3 =	18	0 × 0 =	0

by the value of the resistance of the circuit. Let the lengths of corresponding ordinates of these two curves be multiplied together, and the product set off as the ordinates of a new curve represented by the dotted line in Fig. 33. This dotted curve is, then, the curve of *power* or *activity*, and represents the variation of the product, taken at every instant, of the current and the electromotive force.

In multiplying together the ordinates of the thick and thin curves we must pay attention to the algebraic sign of the factors. Ordinates of each curve drawn above the horizontal datum line of the curve must be reckoned *plus*, and ordinates drawn below must be reckoned *minus*, and in taking the product the algebraic law of signs must be regarded. It will be seen that the dotted line curve consists of a wavy line of two loops lying wholly above the mean datum line. If the area of the two hummocks enclosed by the dotted curve and the horizontal line is *indicated* or integrated, say, by an Amsler's plani-

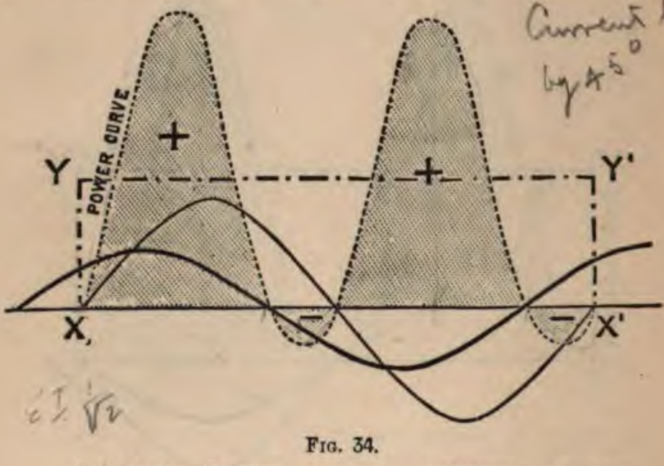

FIG. 34.

ORDINATES.				PRODUCT.
E	×	C	=	P
0	×	2½	=	0
2½	×	3	=	7½
4½	×	3½	=	15½
6	×	3	=	18

ORDINATES.				PRODUCT.
7	×	2½	=	14½
6	×	1	=	6
4½	×	0	=	0
2½	×	−1	=	− 2½
0	×	−2½	=	0

meter, the area represented by the shaded part so obtained is a measure of the total work done in one complete period of the current oscillation, and since this area lies wholly above the datum line, it must be reckoned as *positive*, or as work done by the electromotive force; in other words, it represents the total energy transformed from electrical energy into heat in one complete period.

Next let us suppose that the same periodic electromotive force acts upon a circuit having inductance as well as resistance, and that therefore, as before shown, the current is retarded in phase behind the electromotive force. Let the thin curve in Fig. 34 represent the periodic impressed electromotive force, and the thick curve the current retarded by 45° in phase behind the other. Proceed as before to obtain the power curve by multiplying the heights of corresponding ordinates, the multiplication of the eight ordinates being shown in the margin. We find that the power curve representing the variation of the activity is a wavy curve, consisting of four sections, two large hummocks above the datum line, which are positive areas, and two small ones below,

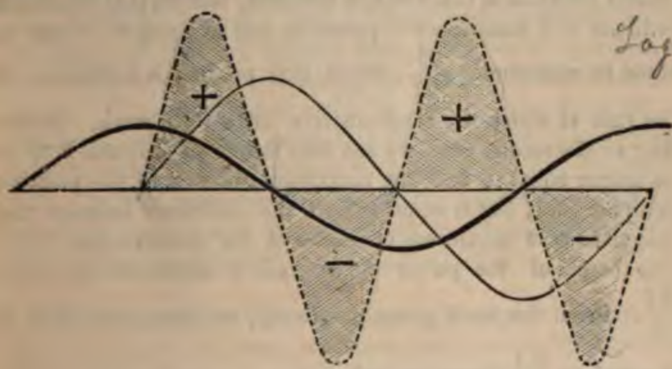

Fig. 35.

Ordinates.				Product.		Ordinates.				Product.
E	×	C	=	P		7	×	0	=	0
0	×	3½	=	0		6	×	−1	=	− 6
2½	×	3	=	7½		4½	×	−2½	=	−11¼
4½	×	2½	=	11¼		2½	×	−3	=	− 7½
6	×	1	=	6		0	×	−3½	=	0

which are negative areas. The algebraic sum taken with regard to sign of all these four areas, represented by the shaded parts, is a measure of the total work done in one complete period by the electromotive force. Going one step more, we may imagine that the current lag is 90°, and in Fig. 35 we have drawn the power curve in this case, obeying the same instructions. We see that the power curve consists of four loops, two positive and two negative, and that the areas of these hummocks are equal. Hence, the total area or *indicated* value in this last case is zero, and the work done in one complete cycle is zero; hence the rate of doing work, or the power, is zero. Returning to Fig. 33, the first case, it is easy to see that the rate of doing work, or the power, is measured by the *mean ordinate* of the shaded work areas considered as indicator diagrams. Since the dotted curves are perfectly symmetrical, if we draw a line Y Y′ at half the height of the maximum ordinate of the dotted curve, it will cut the two hummocks into two parts, and the area of the upper part or mountain above the line Y Y′ would just fill up the valley between the bottom parts of the hummocks. Since the rate of doing work is equal to the work done in one complete cycle divided by the time of duration of that cycle, it obviously follows that this mean ordinate X Y measures the power or rate of doing work and is equal in magnitude to $\frac{EI}{2}$, hence this product is a measure of the rate at which the electromotive force does work. Referring to the second case, we see that the mean ordinate X Y is no longer equal to half the maximum ordinate of the positive or upper loop, but is equal to half the difference between the magnitudes of maximum ordinates of the positive and negative loops of the power curve, and is therefore less than $\frac{EI}{2}$. From the proof given previously, we have seen that it is equal to $\frac{EI}{2}$ (cosine of lag).

In the third case considered, of a lag of 90 degrees, it is easy to see why the resultant rate of doing work is zero. In the first *quarter of a stroke* the electromotive force propels the current, and this last is in the direction of the E.M.F., but in the second quarter of a stroke the current is negative or opposite to the electromotive force; in other words, the current is moving against the force and does work against the E.M.F., and the same push and re-push is repeated in the second half of the period. Hence, on the whole, though there is an impressed electromotive force and a current flowing, no resultant work is done and no energy dissipated. We may construct a mechanical analogy of this as follows:—

Imagine a steam engine with a heavy fly-wheel working without friction. Let steam be applied to it, the slide-valve being so set that it opens when the piston is at the bottom of the cylinder, and cuts off the steam after a very short fraction of a stroke. Suppose the valve then to remain closed for the rest of the stroke, and that instead of opening to exhaust the cylinder at the end of the stroke the valve remains closed, so that the momentum of the moving fly-wheel drives the piston back and compresses the steam back to its original volume. The valve then is supposed to open and permit the steam to escape at the same pressure at which it entered. Assuming no loss of heat (an altogether hypothetical case), we should then have an engine which absorbs at each stroke a cylinder full of steam; but by which no work is being done. The whole operation is a sort of give and take of energy between the fly-wheel and the steam. The same kind of action is taking place in a circuit acted upon by a simple periodic E.M.F. in which the inductive resistance is so large compared with the ohmic that the lag is 90°. In that case there is no frictional dissipation of energy, but simply a give and take of energy between magnetic field and the source of energy supplying the E.M.F.

§ 16. **Experimental Measurement of Periodic Current and E.M.F.**—There are, amongst others, two instruments especially useful for measuring periodic currents. One of these, the Cardew voltmeter, depends upon the principle that when a wire traversed by a current either steady and unidirectional or steadily periodic is placed in an enclosure, the walls of which are approximately at a constant temperature, the wire will itself, after a short time, attain a constant temperature. This constant temperature is reached when there is a state of equilibrium between the rate at which heat is radiated by the wire and the rate at which the walls of the enclosure radiate heat back to it.

The wire has a definite length corresponding to each temperature, and means are provided for measuring this elongation with great accuracy. The total amount of heat generated in the wire per second is dependent upon the rate of generation at each instant. This instantaneous rate of heat development is by Joules's law equal in mechanical units to the product of the ohmic resistance and the square of the instantaneous current strength.

If the wire is traversed by a simple periodic current, and we construct from the current curve diagram another curve, whose ordinates are equal to the square of the corresponding current ordinates, we have a curve every ordinate of which is proportional to the instantaneous rate of generation of heat in a wire traversed by the periodic current. Since the horizontal line measures time, it is obvious that the whole area of the outer curve, or heat curve, represents the total work done per semi-period by the current in producing heat, and that the same total work would be done by a steady current whose value was the square root of the mean of the squares of all the ordinates of the periodic current curve.

This square root of the mean of the squares of all the ordinates of a simple periodic curve has, however, been shown in § 6 to be numerically equal to the value of the maximum ordinate of the periodic curve divided by $\sqrt{2}$. It follows that the total heat generated per second in the wire is a numerical measure of the half the square of the maximum value of a simple periodic current. A fine wire stretched out in the manner of a Cardew voltmeter wire has a very small inductance, and, when acted upon by a simple periodic electromotive force, the current produced in it is very nearly proportional to this impressed electromotive force. It follows then that when a Cardew voltmeter is subjected to a simple periodic E.M.F. the needle takes a definite position, corresponding to a definite expansion of the wire, which is that which it would take if the wire were subjected to a steady electromotive force equal to $\frac{1}{\sqrt{2}}$ of the maximum value of the periodic electromotive force.

The Cardew voltmeter is not adapted to measure any but very small currents. The instrument generally employed to measure periodic currents of moderate and large magnitude is some modification of Weber's electro-dynamometer. In the best-known practical form of Siemens there are two coils of wires in series, one fixed and the other movable, and so placed that the currents in the movable coil circuit are traversed at right angles by the lines of force due to those in the fixed coil. When a simple periodic current traverses the coils in series, a force is brought into existence due to the electro-dynamic action, and which is proportional to the instantaneous value of the square of the current strength. From instant to instant, however, the current strength varies. If the time of free vibration of the moveable coil is very large compared with that of a complete period of the electrical vibrations, and if the movable coil is brought back by a restoring force due to a spring or bifilar suspension or gravity, &c., into a fixed normal position, then, during one complete electrical period, we may consider that the movable portion receives a number of small impulses which are in magnitude represented by the square of the ordinates of the current wave. Hence, the total impulse on the movable coil is equal to the magnitude of the integrated area of a sine curve whose ordinates are respectively the squares of those of the current curve, and the mean force on the movable coil will obviously be proportional to the mean ordinate of this force curve. If the movable coil is so heavy that its time of free vibration is very long compared with the time in which the periodic forces on it run through a complete cycle, it will experience a displacement exactly that due to the mean of the forces acting upon it—that is, to the square root of the mean of the squares of these instantaneous currents—or to $\frac{I}{\sqrt{2}}$, where I is the maximum value of the current during the period. The periodic force on the movable coil is equivalent to a steady force, when this periodic force runs through all its values in a time very short compared with the time of free vibration of the coil. Hence, if a simple periodic current has a maximum value I, when it is sent through an electro-dynamometer it will cause a deflection equal to that which would be caused by a steady current equal to $\frac{I}{\sqrt{2}}$.

Let us suppose a coil of constant inductance L and resist ance R to be traversed by a simple periodic current of frequency n (where $2\pi n = p$). Let an electro-dynamometer be inserted in series with it, and let a Cardew voltmeter be connected to the extremities of the inductive circuit.

We have before seen that if E and I are the maximum values during the period of the impressed E.M.F. and current in an inductive circuit, then the *power* taken up in that circuit is equal to $\frac{E I}{2}\cos\theta$, where θ = angle of lag of current behind the E.M.F. But the reading of the Cardew voltmeter when connected to the ends of an inductive circuit is very nearly proportional to $\frac{E}{\sqrt{2}}$, and the dynamometer reading in that circuit is proportional to $\frac{I}{\sqrt{2}}$; therefore, the product of these readings is proportional to $\frac{E I}{2}$, and takes no account of the difference of phase. For this reason we can derive no information from the use, in this manner, of these instruments. The observed readings, and hence their product, does not take into account the difference of phase between the current and impressed E.M.F. in the inductive circuit. A very small error, in practice negligible, is also introduced by disregarding the inductance of the wire of the Cardew instrument. On this account, strictly speaking, currents in the wire cannot be taken as accurately proportional to potential differences at the extremities, but this is in ordinary usage a negligible error.

§ **17. The Wattmeter.**—If a current traversing an inductive circuit under a periodic impressed electromotive force is made to pass through another circuit which acts electro-dynamically upon a movable circuit conveying another current proportional in strength to, and agreeing in phase with, the periodic variation of potential difference at the terminals of the inductive circuit, such an arrangement will, if it can be realised, afford a means for obtaining a numerical measure of the power taken up in the inductive circuit.

An electro-dynamometer having its fixed coil composed of thick wire and its movable coil of fine wire, each circuit being independent, is most usually called a *wattmeter*. The examination of the circumstances under which the wattmeter can and cannot be used to measure the *power* expended in a circuit subject to simple periodic electromotive force, leads to some very interesting questions.

If the thick and thin wire coils of a wattmeter are traversed by two independent steady unidirectional currents the force on the movable coil is at any instant proportional to the product of the strengths of these two currents. If each of these currents are simple periodic currents the force varies with the product of the instantaneous values, and the compound curve formed by taking as ordinates the products of the corresponding values of these separate current strengths at each instant is itself a simple periodic curve, provided that the two component currents have constant amplitudes, equal period, and

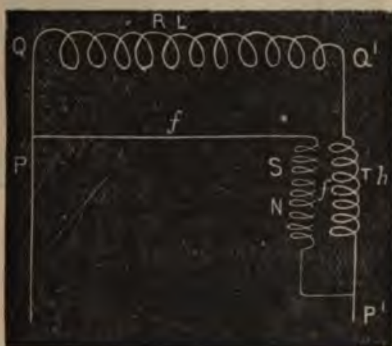

FIG. 36.

fixed difference of phase. Let a wattmeter be supposed to be joined up to an inductive circuit (Fig. 36); let R and L be the resistance and inductance of this inductive circuit between the points Q Q'; let the thick wire coil Th of the wattmeter be joined in series with this inductive resistance, and let the fine wire coil *f* of the wattmeter of resistance S and inductance N be joined to the points P P'; let the thick wire coil be of negligible resistance and inductance in comparison with the circuit Q Q'. If a simple periodic electromotive force operates on the double circuit between the points P and P', we shall have a current flowing in R and S. It is required to calculate at any instant the currents in R and S respectively. Consider simply a divided circuit

(Fig. 37) in which R and S are the branches; let x be the current at any instant in R, and y that in S, and let i be the strength of the current in that part of the circuit just before it divides; in other words, i is the main current, which is divided into x in the inductive resistance, and y in the fine wire coil of the voltmeter. Let e be the potential difference between the points P P′ at the same instant, and let X, Y, I, and E be the maximum values of all these quantities respectively. We assume that i is a simple periodic function of I, and we then write $i = \mathrm{I} \sin pt$, where $p = 2\pi n$, n being

FIG. 37.

the frequency. Applying the fundamental equation of § **11** (3) (p. 105) to each circuit, we see that

$$\mathrm{L}\frac{dx}{dt} + \mathrm{R}x = e;$$

also

$$\mathrm{N}\frac{dy}{dt} + \mathrm{S}y = e.$$

Accordingly,

$$\mathrm{L}\frac{dx}{dt} + \mathrm{R}x = \mathrm{N}\frac{dy}{dt} + \mathrm{S}y;$$

but, by the principle of continuity,

$$i = x + y$$

always, since there can be no accumulation of electricity at P or P′;

hence

$$x = i - y,$$

and, hence,

$$\mathrm{L}\frac{d(i-y)}{dt} + \mathrm{R}(i-y) = \mathrm{N}\frac{dy}{dt} + \mathrm{S}y,$$

or $$L\frac{di}{dt}+Ri=(L+N)\frac{dy}{dt}+(R+S)y.$$

But $$\frac{di}{dt}=I p \cos pt,$$

and $$i=I \sin pt.$$

$$\therefore \quad (L+N)\frac{dy}{dt}+(R+S)y=I L p \cos pt+I R \sin pt,$$

or $$\frac{dy}{dt}+\frac{R+S}{L+N}y=\frac{LpI}{L+N}\cos pt+\frac{RI}{L+N}\sin pt.$$

This differential equation is of the type

$$\frac{dy}{dt}+Py=Q,$$

where Q is a function of t. The solution of this will be found in "Boole's Differential Equations," p. 38, and it is

$$y=e^{-Pt}\left(\int Q e^{Pt} dt+\text{const}\right),$$

e being here the base of Nap. logs. and not impressed E.M.F

In the case before us $$P=\frac{R+S}{L+N},$$

and $$Q=\frac{RI}{L+N}\sin pt+\frac{LpI}{L+N}\cos pt.$$

The integral of $e^{Pt}\sin pt$ and $e^{Pt}\cos pt$ are required. They are as follows:—

$$\int e^{Pt}\sin pt=\frac{e^{Pt}(P\sin pt-p\cos pt)}{P^2+p^2},$$

and $$\int e^{Pt}\cos pt=\frac{e^{Pt}(P\cos pt+p\sin pt)}{P^2+p^2};$$

hence it follows that

$$\int e^{Pt}Q\,dt=\int e^{Pt}\frac{RI}{L+N}\sin pt+\int e^{Pt}\frac{LpI}{L+N}\cos pt$$

$$=\frac{RI}{L+N}e^{Pt}\frac{(P\sin pt-p\cos pt)}{P^2+p^2}+\frac{LpI}{L+N}\frac{e^{Pt}(P\cos pt+p\sin pt)}{P^2+p^2}$$

and $$P^2+p^2=\frac{(R+S)^2+(L+N)^2p^2}{(L+N)^2}.$$

Therefore we have by substitution

$$y=\frac{I}{(R+S)^2+(L+N)^2p^2}\left\{\begin{array}{l}[R(R+S)+L(L+N)p^2]\sin pt\\+[(R+S)Lp-R(L+N)p]\cos pt\end{array}\right\} \quad (7)$$

or

$$y=\frac{I}{(R+S)^2+(L+N)^2p^2}\left\{\begin{array}{r}[R^2+p^2L^2+RS+LNp^2]\sin pt\\+[SLp-RNp]\cos pt\end{array}\right\}.$$

Since the original equations are symmetrical in x and y, R and S, L and N, the value for x is given by changing R to S and L to N in the equation for y.

FIG. 38.

This equation for y gives us the strength of the current in the fine wire coil, and it shows us that the phase of the currents x and y in the branch circuits differs from that of the main current i by an amount which depends on L, N, R and S. In order to exhibit this in a simple form we may direct attention to a simple trigonometrical transformation.

Lemma.—The function $A\sin\theta+B\cos\theta$, where A and B are constants, may otherwise be written

$$\sqrt{A^2+B^2}\sin(\theta+\phi),$$

where $$\tan\phi=\frac{B}{A}.$$

Draw any rectangle (Fig. 38) O P, O Q, and draw a pair of rectangular axes, O X, O Y, through O. Project the points Q, R, P on O Y. Then, by geometry, if $P O X = \theta$ and $P O R = \phi$,

$$O r = O p + O q,$$
$$= O P \sin\theta + O Q \cos\theta,$$
$$= O R \sin R O X = O R \sin(\theta + \phi);$$

hence $$O P \sin\theta + O Q \cos\theta = O R \sin(\theta + \phi)$$
$$= \sqrt{O P^2 + O Q^2} \sin(\theta + \phi).$$

But $$\tan\theta = \frac{O Q}{O P};$$

hence $$A \sin\theta + B \cos\theta = \sqrt{A^2 + B^2} \sin(\theta + \phi) \quad . \; . \; . \; (8)$$

where $$\tan\theta = \frac{B}{A}.$$

Returning then to the equation for y, the coefficients of $\sin p t$ and $\cos p t$ in the equation are respectively $R(R+S) + L(L+N)p^2$, which represents the A, and $(R+S) L p - R (L+N) p$, which represents the B, in the above. Squaring each of these expressions, and adding the results, we obtain as a result

$$\{(R+S)^2 + (L+N)^2 p^2\} (R^2 + p^2 L^2);$$

hence we finally arrive by substitution at the equation for y

$$y = \frac{I \sqrt{R^2 + p^2 L^2}}{\sqrt{(R+S)^2 + p^2 (L+N)^2}} \sin(p t + \theta). \quad . \; (9)$$

where $$\tan\theta = \frac{B}{A} = \frac{(R+S) L p - R (L+N) p}{R (R+S) + L (L+N) p^2}, \quad . \; (10)$$

or, $$\tan\theta = \frac{(S L - R N) p}{R (R+S) + L (L+N) p^2}.$$

In this form the equation for y shows us that the phase of y is *ahead* of that of i, or that the main current lags behind the current in the branch S, provided that S L is greater than R N; and since the expression for the current x is perfectly symmetrical, we can write it down at once, and it is

$$x = \frac{I \sqrt{S^2 + p^2 N^2}}{\sqrt{(R+S)^2 + p^2 (L+N)^2}} \sin(p t + \theta'). \quad . \; (11)$$

where $$\tan \theta' = \frac{(\mathrm{R\,N} - \mathrm{S\,L})p}{\mathrm{S}(\mathrm{R}+\mathrm{S}) + \mathrm{N}(\mathrm{L}+\mathrm{N})p^2}; \quad . \quad . \quad . \quad (12)$$

and it is obvious that if S L is greater than R N, $\tan\theta$ is positive, and $\tan\theta'$ is negative.

If S L = R N, then there is no lag, and the branch currents x and y agree in phase with the main current i.

The general result is, therefore, this—When an impressed electromotive force acts on a circuit which branches into two, having each self but no mutual induction, the main current lags behind the impressed E.M.F. in phase, and the two branch currents respectively *lag behind* and are *pressed ahead* of the phase of the main current.

The question then arises, under what circumstances does the branch current which is in advance in phase of the main current get so much ahead that it comes into consonance with the phase of the impressed E.M.F.?

To settle this question, we shall have to discuss briefly the question of the compound impedance of branch circuits.

§ 18. **Impedance of Branched Circuits.**—Lord Rayleigh has treated the problem of the impedance of branched circuits under the assumption that any number of circuits are connected in parallel, possessing each self-induction, but having no mutual induction.*

The problem is: Given the resistance and induction of each branch, to find the compound resistance and inductance, or equivalent resistance and inductance of the system, for simple periodic currents of given frequency.

Let R and L be the resistance and inductance of any branch, and p the pulsation $= 2\,\pi\,n$. Let R′ and L′ be the compound or equivalent resistance and inductance of the system of parallel conductors.

The solution of the problem given, for which we refer the reader to the original paper, is

$$\mathrm{R}' = \frac{\mathrm{A}}{\mathrm{A}^2 + p^2\,\mathrm{B}^2} \text{ and } \mathrm{L}' = \frac{\mathrm{B}}{\mathrm{A}^2 + p^2\,\mathrm{B}^2},$$

* See Lord Rayleigh "On Forced Harmonic Oscillations of Various Periods" (*Phil. Mag.*, May, 1886, p. 379).

where $$A = \Sigma\left(\frac{R}{R^2 + p^2 L^2}\right)$$

and $$B = \Sigma\left(\frac{L}{R^2 + p^2 L^2}\right).$$

If we take, as usual, $\tan\theta = \frac{pL}{R}$, and write (Im) for impedance, where $(\mathrm{Im})^2 = R^2 + p^2 L^2$, we can write the above relations

$$A = \Sigma \frac{R}{(\mathrm{Im})^2}$$

$$B = \Sigma \frac{L}{(\mathrm{Im})^2}.$$

Let $R'^2 + p^2 L'^2$ be written $(\mathrm{IM})^2$. This is the compound or

FIG. 39.

equivalent impedance of the system of parallel conductors. It is obvious that

$$(\mathrm{IM})^2 = R'^2 + p^2 L'^2 = \frac{1}{A^2 + p^2 B^2},$$

where $$A = \Sigma \frac{R}{(\mathrm{Im})^2} \text{ and } B = \Sigma \frac{L}{(\mathrm{Im})^2};$$

hence $$(\mathrm{IM})^2 = \frac{1}{\left(\Sigma \frac{R}{(\mathrm{Im})^2}\right)^2 + \left(\Sigma \frac{L}{(\mathrm{Im})^2}\right)^2}.$$

Consider the case of a pair of conductors in parallel (Fig. 39), having resistances R and S and inductances L and N, but no mutual inductance.

Let
$$\sqrt{R^2+p^2 L^2}=(\mathrm{Im}_1),$$
and
$$\sqrt{S^2+p^2 N^2}=(\mathrm{Im}_2),$$
and
$$\sqrt{R'^2+p^2 L'^2}=(\mathrm{IM});$$
then
$$(\mathrm{IM})^2=\frac{1}{\left(\frac{R}{(\mathrm{Im}_1)^2}+\frac{S}{(\mathrm{Im}_2)^2}\right)^2+\left(\frac{pL}{(\mathrm{Im}_1)^2}+\frac{pN}{(\mathrm{Im}_2)^2}\right)^2};$$
or
$$(\mathrm{IM})=\frac{(\mathrm{Im}_1)\,(\mathrm{Im}_2)}{\sqrt{(\mathrm{Im}_1)^2+(\mathrm{Im}_2)^2+2\,(R\,S+p^2\,L\,N)}};$$
or
$$(\mathrm{IM})=\frac{\sqrt{R^2+p^2 L^2}\,\sqrt{S^2+p^2 N^2}}{\sqrt{(R+S)^2+p^2\,(L+N)^2}}.$$

The lag ϵ of the main current just before branching, considered with respect to the impressed electromotive force, will be given by the equation
$$\tan\epsilon=\frac{p\,L'}{R'};$$
hence
$$\tan\epsilon=\frac{p\,B}{A}=\frac{\Sigma\frac{pL}{(\mathrm{Im})}}{\Sigma\frac{R}{(\mathrm{Im})}}$$
generally, and in the case considered will be
$$\tan\epsilon=\frac{\frac{pL}{R^2+p^2L^2}+\frac{pN}{S^2+p^2N^2}}{\frac{R}{R^2+p^2L^2}+\frac{S}{S^2+p^2N^2}};$$
hence after reduction
$$\tan\epsilon=\frac{(S^2+p^2N^2)\,p\,L+(R^2+p^2L^2)\,p\,N}{(S^2+p^2N^2)\,R+(R^2+p^2L^2)\,S}.$$
This is the equation which determines the *lag of phase* of the current i behind the impressed electromotive force in the main branch before dividing into the branch currents x and y in R and S respectively.

Compare this equation with that which determines the angle by which the phase of the branch current y in S is *ahead* of the main current i. It is, as we have seen,
$$\tan\theta=\frac{(S\,L-R\,N)\,p}{R\,(R+S)+L\,(L+N)\,p^2}.$$

In the expressions for tan ϵ and tan θ put $N=0$, and they *both* become equal to

$$\frac{S L p}{R(R+S)+p^2 L^2}.$$

This shows that, when $N=0$, the current y in the branch S is as much ahead of the main current i as i is behind the impressed electromotive force, and hence that y agrees in phase with the impressed E.M.F. acting on the double circuit; in other words, the current in the branch S is entirely unaffected by being joined in parallel with an inductive circuit R; but if N is *not* quite zero, then the current in branch S *is* affected, as regards its lag, by the fact of being joined in parallel with an inductive circuit. The nature of this affection will be dependent on whether $SL-RN$ is positive or negative—that is, whether $\frac{L}{R}$ or $\frac{N}{S}$ is the greater—that is, whether the time-constant of the R circuit or the S circuit is greater. If $\frac{L}{R}$ is greater than $\frac{N}{S}$, then the current y in S is *ahead* of the main current i, but lags behind the impressed electromotive force. If $\frac{L}{R}$ is less than $\frac{N}{S}$, then the current y in S lags behind the main current i in phase, and, *à fortiori*, behind the impressed electromotive force.

§ 19. **Wattmeter Measurement of Periodic Power.**—Returning to the wattmeter problem, let one of these divided circuits, viz., the one of resistance R, be a circuit in which it is desired to measure the *electrical power*. In the ordinary way of using the wattmeter the fine-wire coil, which we will assume has a resistance S, is placed in parallel with the inductive circuit, the thick-wire coil united in series with the inductive circuit. The main current i is thus divided between the inductive circuit R and the wattmeter fine-wire circuit S. The electro-dynamic action in the wattmeter is then one between a current in S, which we have called y, and one in the thick-wire circuit, which is the same as that in the inductive circuit R, which we have called x.

We have above arrived at expressions for the values of x

and y. The question then arises how far the indications given by the instrument, and which are due to the electro-dynamic action of the currents x and y, and proportional to their numerical product, are proportional to the real power taken up in the circuit R.

The current x is the same as the current in R; hence the error, if any, will result from the current y in S differing in phase or in proportionality from the potential difference between the ends of the circuit R.

In the ordinary mode of calibrating the wattmeter the instrument would be applied to measure the power in a non-inductive circuit traversed by a known current, and having a known potential difference at its ends.

From this the real watts taken up in the circuit are known, and since the force required to bring back the movable coil to its initial position is proportional to the product of the currents in the fixed and movable coils, we have at once the desired constant of the instrument.

If a wattmeter so calibrated is applied to measure power in an inductive circuit, there are two different causes of error which may or may not neutralise each other, and which may cause the measured watts as determined by the instrument to be greater than, equal to, or less than, the real watts or power taken up in the circuit.

The first of these causes of error is due to the fact that the fine-wire circuit of the wattmeter always has a sensible inductance—that is, N is not zero. It is true it may be made very small, and can be made small by the device adopted by some makers of arranging the chief part of the wire resistance of the fine-wire circuit as a non-inductive resistance in series with a small inductive resistance which forms the movable coil. It follows that if E be the maximum potential difference during the period between those points to which the fine-wire circuit is attached, that the true average current in the fine-wire circuit is equal to $\frac{2}{\pi} \cdot \frac{E}{\sqrt{S^2+p^2 N}}$, when subjected to a simple periodic E.M.F. of pulsation p. This quantity is not always proportional to E, but depends on p. The effect of the impedance of the fine-wire circuit is always to make the true mean current in it under periodic E.M.F. *less* than it would be if produced by a steady E.M.F. equal to the true mean of the periodic E.M.F. But, in addition, the impedance causes a lag in phase of the current in the fine wire circuit behind the phase of the potential difference between its ends. This is the second cause of error, and the effect of this lag is dependent upon the nature, whether inductive or non-inductive, of the circuit R. To dissect its action, first let us suppose the circuit R is non-inductive—that is, L is zero. The current x in it will, therefore, coincide in phase with that of the potential difference at the points of junction.

The current in S, viz., y, will, however, lag in phase behind that of the potential difference at the junction. The effect of this lag in S will be to increase the phase difference between x and y, and to diminish the cosine of this angle of phase difference. Hence, to diminish the product $\frac{XY}{2}$, cos δ, which measures the true mean product of x and y, X and Y being their maximum values, and δ their difference of phase. Since by assumption X agrees in phase with E, any reduction of the above product reduces the instrumental reading, and makes it less than the true-power reading. If, however, we have to deal with a circuit possessing inductance, and in which, therefore, there is a current x, of which the phase lags behind that of the potential difference of the junctions, then the lag in the current y in the circuit S, so far from increasing the difference of phase of x and y, may operate to bring them nearer into accordance, and to increase the instrumental reading over and above that decrease due to the first-named cause of error.

§ **20. Correcting Factor of a Wattmeter.**—The action of these two causes of error may be illustrated and explained best by the geometric method.

Describe a circle with centre O (Fig. 40), and take any line O A to represent the maximum value of the potential difference between the two points M M′ of the divided circuit, of which R is the resistance of the inductive circuit consisting of the thick wire of the wattmeter in series with the circuit in which the *power* is being measured, and S that of the fine wire of the wattmeter. Then, as before, the vertical projections of O A as it revolves represent the periodic variation of this potential difference. On O A describe a semi-circle, and set off on O A as a base, two right angle triangles O C A, O B A, of which the sides O B, B A, and O C, C A are in the ratio respectively of the ohmic to the inductive resistances of these circuits. Otherwise the angle A O B is one whose tangent is p times the time-constant of the S circuit, and A O C is one

Fig. 40.

whose tangent is p times the time constant of the R circuit. Take one Sth portion of O B, and set off O Y equal to it, then, as in § **13** (p. 115), O Y represents the maximum value of the current in the S circuit.

Similarly, set off O X equal to one Rth part of O C, and O X represents the maximum current in R. On O X, O Y, describe parallelogram O Y I X, and draw the diagonal O I, and produce it to O D. Then O I represents the maximum current just before division on the main currents. Join AD, AD and OD will represent the compound or equivalent inductive and ohmic resistance of the two circuits R and S in parallel.

To prove this last proposition, we must refer again to the paper by Lord Rayleigh on "Forced Harmonic Oscillation of Various Periods" (*Phil. Mag.*, 1886).

If R′ represents the equivalent resistance of a number of resistances joined in parallel between two points, and L′ represent the equivalent inductance of the system, then it is shown in Lord Rayleigh's paper that

$$R' = \frac{A}{A^2 + p^2 B^2} \text{ and } L' = \frac{B}{A^2 + p^2 B^2},$$

where $$A = \Sigma \frac{R}{R^2 + p^2 L^2},$$

and $$B = \Sigma \frac{L}{R^2 + p^2 L^2},$$

R and L being the resistance and inductance of any branch, and mutual inductance being zero.

Apply this theorem to the case under consideration of the two inductive resistances (R, L) (S, N) in parallel, and we

have $$A = \frac{R}{R^2 + p^2 L^2} + \frac{S}{S^2 + p^2 N^2},$$

$$B = \frac{L}{R^2 + p^2 L^2} + \frac{N}{S^2 + p^2 N^2}.$$

Effecting the multiplication we have

$$A = \frac{R\,(S^2 + p^2 N^2) + S\,(R^2 + p^2 N^2)}{(S^2 + p^2 N^2)\,(R^2 + p^2 L^2)},$$

$$B = \frac{L\,(S^2 + p^2 N^2) + N\,(R^2 + p^2 L^2)}{(S^2 + p^2 N^2)\,(R^2 + p^2 L^2},$$

and $$R' = \frac{A}{A^2 + p^2 B^2} = \frac{R\,(S^2 + p^2 N^2) + S\,(R^2 + p^2 L^2)}{(R+S)^2 + p^2\,(L+N)^2},$$

$$L' = \frac{B}{A^2 + p^2 B^2} = \frac{L\,(S^2 + p^2 N^2) + N\,(R^2 + p^2 L^2)}{(R+S)^2 + p^2\,(L+N)^2}.$$

Turning back to Fig. 40, we see from the geometry of it that, if the angle B O D is as before called θ, B O D = D A B.

Hence $$\frac{\text{O D}}{\cos\theta} = \text{S} - p\,\text{N} \tan\theta,$$

since A B = p N and O B = S by construction,

therefore $$\text{O D} = \text{S} \cos\theta - p\,\text{N} \sin\theta,$$

But in § 16 we have found the value of tan θ to be

$$\tan\theta = \frac{(\text{S L} - \text{R N})\,p}{\text{R}(\text{R}+\text{S}) + \text{L}(\text{L}+\text{N})\,p^2};$$

hence substituting we have

$$\text{O D} = \frac{\text{S}\{\text{R}(\text{R}+\text{S}) + \text{L}(\text{L}+\text{N})\,p^2\} - p^2\,\text{N}(\text{S L} - \text{R N})}{(\text{R}+\text{S})^2 + (\text{L}+\text{N})^2\,p^2},$$

or $$\text{O D} = \frac{\text{R S}(\text{R}+\text{S}) + p^2(\text{S L}^2 + \text{R N}^2)}{(\text{R}+\text{S})^2 + (\text{L}+\text{N})^2\,p^2}.$$

On comparing this value for O D with the value above calculated for R′ we see they are equal, and hence O D = R′.

So that on the same scale on which O B and O C represent S and R, the individual resistances, O D represents the equivalent resistance for the periodicity employed. Similarly, it may be shown that A D = p L′. For the angle C O D = θ' = angle C A D, and

$$\frac{\text{A D}}{\cos\theta'} = p\,\text{L} - \text{R} \tan\theta',$$

or $$\text{A D} = p\,\text{L} \cos\theta' - \text{R} \sin\theta';$$

and since $$\tan\theta' = \frac{(\text{R N} - \text{S L})\,p}{\text{S}(\text{R}+\text{S}) + \text{N}(\text{L}+\text{N})\,p^2},$$

a similar substitution enables us to see that

$$\text{A D} = \frac{p\,\text{L}(\text{S}^2 + p^2\,\text{N}^2) + p\,\text{N}(\text{R}^2 + p^2\,\text{L}^2)}{(\text{R}+\text{S})^2 + (\text{L}+\text{N})^2\,p^2},$$

and this is equal to the value found by analysis for p L′.

The diagram also shows us many other things besides this construction for the equivalent resistance and inductance.

It shows us, for instance, precisely what is the effect of the inductance of the wattmeter fine-wire circuit, and what must be the correction applied to the readings to get the real power expended in the inductive circuit.

The actual reading of the wattmeter is proportional to the true mean value of the product of x the current in the inductive circuit R, and y the current in the fine-wire circuit S; and this as previously shown is equal to half the product of their maximum values, and the cosine of the difference of phase.

From Fig. 40 (p. 140) this mean value is therefore

$$\frac{OX \, . \, OY}{2} \text{ cosine } BOC.$$

This, however, is *not* the measure of the power expended in the R circuit. The true watts are proportional to the mean product of x and a current equal to one S^{th} part of e, having a phase difference equal to the angle C O A, viz., that of the angle of lag of the current in R and the potential difference O A of its ends; hence the real power or watts are proportional

to $$\tfrac{1}{2} \frac{E}{S} \, . \, OX \, . \text{ cosine } COA,$$

since E is the maximum of e, viz., the instantaneous potential difference between the extremities of the branch circuits.

Now O Y is taken as one S^{th} part of the effective electromotive force in the S circuit; and on the same scale, on which O A represents the impressed E.M.F., O B represents the effective E.M.F. in that circuit. Hence, in taking the reading of the wattmeter, which is proportional to the quantity

$$\frac{OX \, . \, OY}{2} \text{ cosine } BOC$$

as the watts, we are making an error; the quantity really required is the value of

$$\frac{1}{2}\frac{E}{S} OX \text{ cosine } AOC,$$

which is numerically equal to the real power. We see that two errors come in—one due to the maximum current in the fine-wire circuit being O Y or $\frac{OB}{S}$ instead of $\frac{E}{S}$ or $\frac{OA}{S}$ and the other due to the phase difference being taken as the angle C O B instead of C A O.

To correct the instrumental reading or observed watts to true value or real watts, we have to multiply the observed readings by two factors.

First, the ratio of $\frac{OA}{OB}$ or $\frac{\sqrt{S^2+p^2N^2}}{S}$,

which is the correction due to the self-induction of the fine wire circuit or to the potential part of the wattmeter having a sensible inductance. The *second* is the ratio of the cosines of the angles C O A and C O B, or

$$\frac{\cos COA}{\cos COB} = \frac{\cos COA}{\cos(COA - BOA)} = k,$$

but from the diagram

$$\text{cosine } COA = \frac{R}{\sqrt{R^2+p^2L^2}},$$

and

$$\text{cosine } BOA = \frac{S}{\sqrt{S^2+p^2N^2}}$$

$$\therefore k = \frac{\frac{R}{\sqrt{R^2+p^2L^2}}}{\frac{R}{\sqrt{R^2+p^2L^2}} \cdot \frac{S}{\sqrt{R^2+p^2L^2}} + \frac{pL}{\sqrt{R^2+p^2+L^2}} \cdot \frac{pN}{\sqrt{S^2+p^2N^2}}}.$$

Combining these two corrections into a single product, we get as the full correcting factor :—

$$\frac{\sqrt{S^2+p^2N^2}}{S} \cdot \frac{R\sqrt{S^2+p^2N^2}}{RS+p^2LN},$$

or

$$\frac{RS^2+p^2N^2R}{RS^2+p^2LNS} = F.$$

If we put $T_S = \frac{N}{S}$ where T_S is the time-constant of the S or fine-wire circuit, and $T_R = \frac{L}{R}$ where T_R is the time-constant of the R circuit, we have

$$F = \frac{1 + p^2 T_S^2}{1 + p^2 T_R T_S},$$

and the real watts or power taken up in the circuit R is obtained by multiplying the observed watts by F. F becomes unity for two cases when L and N are both zero, and also when $T_S = T_R$.

Hence, the ordinary wattmeter, applied as usual to measure the *electrical power* in a circuit traversed by a simple periodic current, gives absolutely correct readings only in two cases. First, when the fine-wire circuit and the circuit being measured have no inductance; second, when the fine-wire circuit and the circuit being measured have equal time-constants.

But if T_R is greater than T_S, then F is a proper fraction. The wattmeter reads too high, and the real watts are less than the observed. If T_R is less than T_S, then the observed readings are too low. If $T_R = T_S$, then the observed readings are correct. Hence the wattmeter may read too high, too low, or correct. Generally speaking, it reads too high, since the time-constant of the measured circuit will most often be in excess of that of the fine-wire circuit.

§ 21. **Mutual Induction of Two Circuits of Constant Inductance.**—We will conclude this chapter on the theory of simple periodic currents moving in circuits of constant inductance by examining the mutual induction between two circuits in one of which a simple periodic electromotive force operates. We suppose two circuits to be so placed relatively to each other that when a change of current occurs in one, which is called the Primary (Pr.), a change of magnetic induction takes place through the other, called the Secondary (Sec.). We have, then, to regard the primary and secondary as linked together by loops of induction, and the closed lines of induction, together with the two circuits, must be considered as forming three links of a chain. We shall suppose the self and mutual inductance to be known, as also the resistance of each circuit. The primary inductance and resistance will be denoted by L and R, and those of the secondary by N and S, and the mutual inductance by M. The primary is to be subjected to a simple periodic electromotive force of which the maximum value is E, and the result is to generate in the primary circuit a primary current, which, as we have seen, is also a simple periodic quantity, and is to be denoted by its maximum value, I_1. The change of induction through the secondary follows the change of current, and gives rise to an impressed electromotive force in the secondary circuit, which, being represented by the rate of change of the simple periodic induction, is also a simple periodic quantity, and gives rise to a simple periodic current in the secondary, to be denoted by its maximum value I_2.

The general description of the phenomena produced in such a system of primary and secondary circuits connected by an air magnetic circuit is as follows :—

1. The application of a simple periodic impressed electromotive force, E, produces a simple periodic current, I_1, moving under an effective electromotive force, $R\,I_1$, and brings into existence a counter electromotive force of self-induction, which causes the primary current I_1 to lag behind E by an angle called the primary lag θ_1. If n is the frequency of the vibrations and $2\pi n = p$, as before, then, as we have before seen, $\tan\theta_1 = \frac{Lp}{R}$, and this counter electromotive force of self induction is a periodic quantity of which the maximum is $Lp\,I_1$, and of which the phase is 90° behind that of the effective electromotive force or current, or is in quadrature with it.*

2. The field round the primary, and therefore the induction through the secondary, is in consonance with the primary current I_1; but, since it is also a simple periodic quantity, its time-rate of change, and therefore the impressed electromotive force in the secondary, is in quadrature with the primary current. Since the induction through the secondary, due to

* Mr. T. H. Blakesley has employed the term *in quadrature* to express the fact that one simple periodic quantity lags 90° behind another ; hence the electromotive force of self-induction is said to be *in quadrature* with the effective electromotive force or current.—*See* T. H. Blakesley, "On Magnetic Lag," *Electrician*, May 25, 1888, p. 88 ; *Phil. Mag.*, July, 1888.

a current I_1 in the primary, is $M I_1$, by the definition of M the maximum value of the rate of change of this induction for a pulsation p is $M p I_1$.

It is useful to note that in all dealings with simple periodic quantities, if X is the maximum value of a simple periodic quantity which runs through its cycle n times in a second, the maximum value of its time-rate of change is denoted by $p X$, where $p = 2 \pi n$.

If, then, as usual, simple periodic quantities are denoted by the letter signifying their maximum values, prefixing p to any one gives us the value of the maximum of its first differential coefficient with regard to time, or p is here equivalent in notation to $\frac{d}{dt}$.

3. This secondary impressed electromotive force gives rise to a secondary current, I_2, moving under an effective secondary electromotive force, $S I_2$, and creating a counter electromotive force of self-induction in the secondary, represented by $N p I_2$.

The secondary current lags behind the secondary impressed electromotive force by an angle θ_2 such that

$$\tan \theta_2 = \frac{N p}{S}.$$

4. This secondary current, I_2, reacts in its turn on the primary, and it creates what is called a back electromotive force, or reacting inductive electromotive force on the primary circuit. The phase of this must be in consonance with that of the electromotive force of self-induction in the secondary, and it is represented by the quantity $M I_2 p$. This is obviously in quadrature with the phase of the secondary current or secondary effective electromotive force.

5. There is, then, a phase difference between the primary and secondary currents, and also between the primary impressed electromotive force and the primary current.

The general problem is, then: Given the value of the inductances L, M, N, and the resistances R, S, and that of the impressed electromotive force E and the frequency n, find from these seven quantities other four, viz., the primary current I_1, the secondary I_2, and the difference of phase between E, I_1 and I_2.

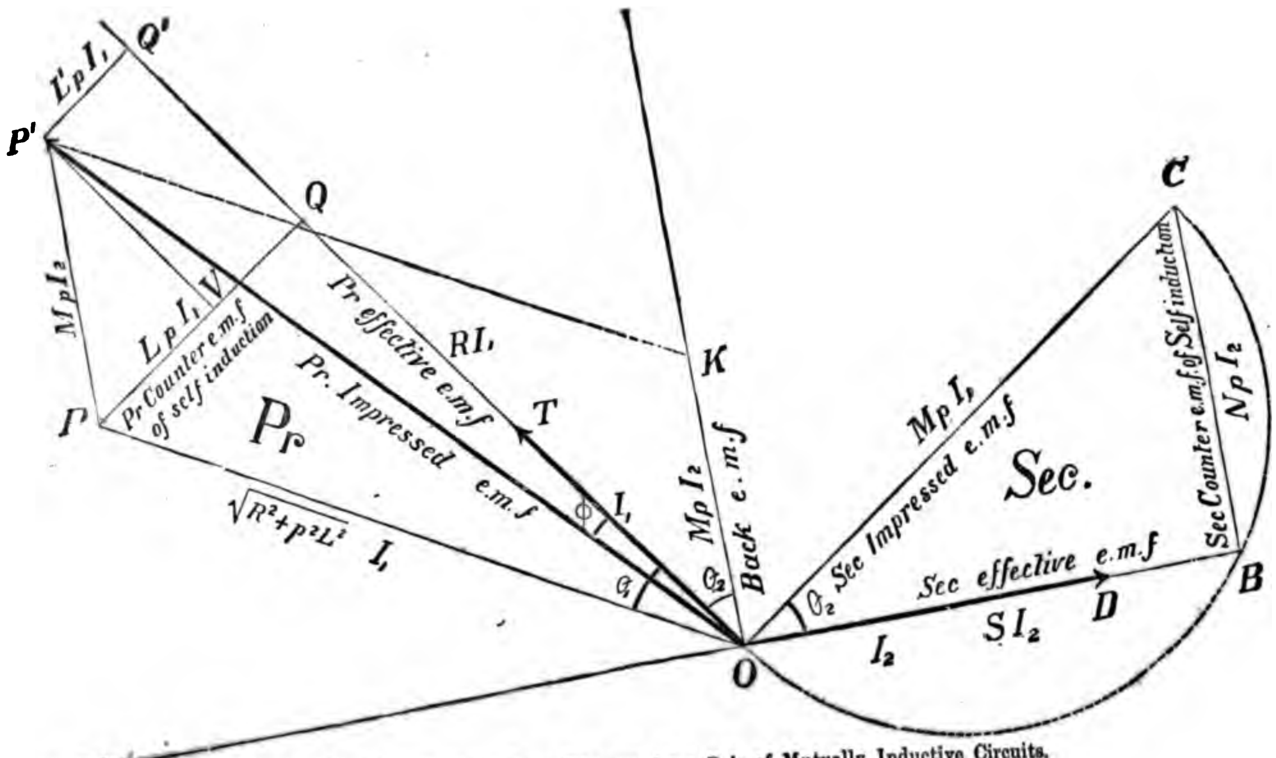

FIG. 41.—Electromotive Force Diagram for a Pair of Mutually Inductive Circuits.

We shall attack the problem geometrically, as this method exhibits far better than the algebraic method the relation between the various quantities involved. The method adopted is to construct an electromotive force diagram, in which all lines represent on any scale volts; and moreover, as each of the quantities considered is a periodic quantity, the lines all represent the maximum value of each quantity, and the value at any instant can be obtained by taking the projections of all lines on any straight line through the centre of the diagram suitably placed.

Let O (Fig. 41) be taken as a centre; draw any line O Q′, and on it set off any length, O T, which we *assume* as the magnitude of the maximum of the primary current. All other lines will be in proper proportion to this. Produce O T to O Q so that O Q $= \mathrm{R}\, \mathrm{I}_1$. O Q is then the effective electromotive in the primary circuit. From Q draw Q P at right angles to O Q, and set off Q P equal to $\mathrm{L}\, p$ times O T or to $\mathrm{L}\, p\, \mathrm{I}_1$; Q P represents the electromotive force of self-induction in the primary circuit. Join O P.

From O draw O C at right angles to O Q, and set off O C equal to $\mathrm{M}\, p$ times O T. O C is then equal to $\mathrm{M}\, p\, \mathrm{I}_1$, and O C represents the impressed electromotive force in the secondary circuit. On O C describe a semi-circle, and set off O B, making an angle C O B with O C such that tan C O B $= \frac{\mathrm{N}\, p}{\mathrm{S}}$, or tan C O B = the ratio of the inductive to the ohmic resistance for the secondary circuit. Join B C. On the same scale on which O C represents the impressed electromotive force in the secondary circuit, viz., $\mathrm{M}\, p\, \mathrm{I}_2$, O B will then represent the effective electromotive force in the secondary, or will represent $\mathrm{S}\, \mathrm{I}_2$, and hence, if O D is taken equal to one S^{th} part of O B, O D will represent I_2, or the secondary current. Next draw a line O K perpendicular to O B, and therefore parallel to B C, and on it set off a length, O K, equal to $\mathrm{M}\, p$ times O D or to $\mathrm{M}\, p\, \mathrm{I}_2$. O K represents then the back inductive electromotive force of the secondary on the primary.

The impressed electromotive force which has to be applied to the primary to produce in it the primary current O T and to induce in the secondary the secondary current O D has therefore to be equal and opposite to the resultant of three electromotive forces, or to equilibrate three electromotive forces, viz., the effective electromotive force of the primary O Q, the electromotive force of self-induction in the primary P Q, and the back electromotive force in the primary due to the inductive effect of the secondary on the primary, viz., O K.

The resultant of O Q and Q P is O P. If, then, we draw P P′ from P, and make it equal and parallel to O K, and join O P′, O P′ will be the resultant of O Q, Q P, and O K; and hence O P′ will represent E, or the impressed electromotive force required to be applied to the primary to maintain the currents I_1 and I_2.

It is to be understood that in this diagram a unit of length stands for a volt, an ampere, an ohm, and a secohm; and hence, on that assumption, E represents in volts the impressed E.M.F.—that is, the maximum of the simple periodic E.M.F. required to maintain the currents I_1 I_2, of which O T, O D represent the maximum values. The relative phases are indicated by the positions of these lines. To obtain the actual values of the E.M.F. and currents at any instant we have only to take the projections of O P′, O T, and O D on any line drawn through O suitably placed, and the magnitudes of these projections will give the required quantities. We must then suppose the whole diagram to be enlarged or diminished without distortion until the length of O P′ is numerically equal to the maximum value in volts of the impressed E.M.F. E, and then O T and O D will represent the currents I_1 and I_2 in magnitude. We may consider the two right-angled triangles O Q P, O B C as pivoted together at O, and revolving round O; the fluctuations of the projections of O P′, O T, O D on any line will give us the cyclic values of E, I_1, and I_2. We can next obtain some useful relations between these quantities from the geometry of the figure. In the triangle O B C,

$$OC^2 = OB^2 + BC^2.$$

Hence $$I_1^2 M^2 p^2 = S^2 I_2^2 + N^2 p^2 I_2^2;$$

or $$\frac{I_1}{I_2} = \frac{\sqrt{S^2 + p^2 N^2}}{Mp};$$

or $$\frac{\text{primary current}}{\text{secondary current}} = \frac{\text{impedance of secondary}}{Mp}.$$

Mp might by analogy be called the mutual inductive resistance.

To obtain the value of I_1 in terms of E, and the inductances and resistances, we project the lines O P′ and O P on the vertical line O K, and express the fact that O P′ or E is in all cases the resultant of O K and O P. Let the angle P′ O Q′ be called ϕ. ϕ is the angle by which the primary current lags behind the total impressed electromotive force. Then C O B is θ_2, and T O K = C O B = θ_2, since Q O C and K O B are both right angles.

Hence we have by resolution on O K

$$E \cos(\phi+\theta_2) = M p I_2 + \sqrt{R^2+p^2 L^2}\, I_1 \cos(\theta_1+\theta_2);$$

but since

$$I_2 = I_1 \frac{M p}{\sqrt{S^2+p^2 N^2}},$$

we have by substitution

$$E \cos(\phi+\theta_2) = \left\{ \frac{M^2 p^2}{\sqrt{S^2+p^2 N^2}} + \sqrt{R^2+p^2 L^2} \cos(\theta_1+\theta_2 \right\} I_1,$$

a relation established between E and I_1 which is known when ϕ is known.

Since

$$\tan\theta_1 = \frac{pL}{R}, \text{ and } \tan\theta_2 = \frac{pN}{S},$$

it follows by an easy transformation that

$$\cos(\theta_1+\theta_2) = \frac{RS - p^2 LN}{\sqrt{R^2+p^2 L^2}\cdot\sqrt{S^2+p^2 N^2}};$$

hence

$$E = \frac{M^2 p^2 + R^2 - p^2 L N}{\sqrt{S^2+p^2 N^2}} \, \frac{I_1}{\cos(\phi+\theta_2)}.$$

To find the value of ϕ, suppose that whilst E and I_1 remain the same the secondary circuit is suppressed. We should then only have an impressed electromotive force, E, creating a current, I_1, and from the diagram and from what has been before explained it is obvious that the effective and self-inductive electromotive forces in the circuit would then be represented by O Q′ and Q′ P′. If we denote these by the symbols

$R' I_1$ and $L' p I_1$ we may properly call R' and L' the equivalent resistance and inductance; that is to say, these quantities are the resistance and inductance which the primary circuit should have in order that, when the secondary circuit is removed, the primary impressed electromotive force may generate in it the same current which it does when the secondary circuit is present and the primary has its natural resistance R, and inductance L. We see, then, that the effect of bringing up the secondary and allowing it to be acted upon and react upon the primary is to increase the effective resistance and diminish the effective inductance of the primary; in other words, the equivalent resistance of the primary circuit is greater and the equivalent inductance is less by reason of the presence of the secondary circuit.

We have then to find the value of $\cos(\phi + \theta_2)$.

$$\text{Cos}\,(\phi + \theta_2) = \cos\phi \cos\theta_2 - \sin\phi \sin\theta_2;$$

but $$\cos\phi = \frac{R'}{\sqrt{R'^2 + p^2 L'^2}}, \qquad \sin\phi = \frac{L' p}{\sqrt{R'^2 + p^2 L'^2}},$$

and $$\cos\theta_2 = \frac{S}{\sqrt{S^2 + p^2 N^2}}, \qquad \sin\theta_2 = \frac{N p}{\sqrt{S^2 + p^2 N^2}};$$

hence $$\cos(\phi + \theta_2) = \frac{R' S - L' N p^2}{\sqrt{S^2 + p^2 N^2} \quad \sqrt{R'^2 + p^2 L'^2}}.$$

Substituting this in the equation connecting E and I_1, we arrive at

$$E = I_1 \frac{(M^2 p^2 + R S - p^2 L N) \quad \sqrt{R'^2 + p^2 L'^2}}{R' S - L' N p^2}.$$

Returning to Fig. 41, we see from it that P′ Q′ is parallel to P Q, and hence, if we draw P′ V parallel to Q′ Q, we have

$$\begin{aligned} P' Q' &= P Q - P V \\ &= P Q - P P' \cos P' P V \\ &= P Q - P P' \sin\theta_2, \end{aligned}$$

or $$L' p I_1 = L p I_1 - M p I_2 \frac{N p I_2}{M p I_1};$$

or, since $$\frac{I_2^2}{I_1^2} = \frac{M^2 p^2}{S^2 + p^2 N},$$

we have $$L' = L - \frac{N M^2 p^2}{S^2 + p^2 N^2} \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Also, again, O Q represents $R\,I_1$ and O Q′ on the same scale $R'\,I_1$, and

$$\begin{aligned} O\,Q' &= O\,Q + Q\,Q' \\ &= O\,Q + P'\,V \\ &= O\,Q + P\,P' \sin P'\,P\,V \\ &= O\,Q + P\,P' \cos\theta_2; \end{aligned}$$

hence $$R'\,I_1 = R\,I_1 + M\,p\,I_2 \frac{S\,I_2}{M\,p\,I_1};$$

and therefore $$R' = R + \frac{M^2 p^2 S}{S^2 + p^2 N^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

These formulæ (A) and (B) give us the effective inductance and resistance of the primary circuit as affected by the secondary. They were first given by Clerk Maxwell in a paper in the *Philosophical Transactions* of the Royal Society in 1865, entitled "A Dynamical Theory of the Electromagnetic Field" (*Phil. Trans.*, 1865, p. 475).*

If we form from (A) and (B) the function $R'S - L'Np$, we find it to be $M^2p^2 + RS - p^2LN$; and hence, by substitution in the expression already given, connecting E and I_1, we arrive finally at the result

$$E = I_1 \sqrt{R'^2 + p^2 L'^2},$$

or $$I = \frac{E}{\sqrt{R'^2 + p^2 L'^2}}.$$

Following the usual nomenclature, we may call the expression $\sqrt{R'^2 + p^2 L'^2}$ the *equivalent impedance* of the primary circuit, and we have as the final result for the induction coil of constant inductance

$$\text{Primary current strength} = \frac{\text{impressed electromotive force}}{\text{equivalent impedance of primary circuit}},$$

$$\text{Secondary current strength} = M\,p \times \frac{\text{primary current}}{\text{impedance of secondary circuit}}.$$

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* *See also* Lord Rayleigh on "Forced Harmonic Oscillations of Various Periods," *Phil. Mag.*, May, 1886, p. 375.

The angle of lag of primary current behind impressed E.M.F $= \phi$ $\tan\phi = \frac{pL'}{R'}$, and the angle of lag of the secondary current behind the primary is seen to be $90° + \theta_2$ and $\tan\theta_2 = \frac{Np}{S}$; hence we have the values and relative phases of the currents and the impressed electromotive force.

In the above equations we are to understand current strengths and electromotive forces to be either the maximum values or the average values during the period. If i_1 and i_2 be the actual values at any time t, reckoning time from the instant of the zero value of the electromotive force, then, from the principles previously explained in this chapter, it is obvious that

$$i_1 = I_1 \sin(pt - \phi),$$

or $$i_1 = \frac{E}{\sqrt{R'^2 + p^2 L'^2}} \sin(pt - \phi),$$

and $$i_2 = \frac{Mp\,i_1}{\sqrt{S^2 + p^2 N^2}};$$

or $$i_2 = \frac{Mp}{\sqrt{S^2 + p^2 N^2}} \frac{E}{\sqrt{R'^2 + p^2 L'^2}} \sin(pt + \phi + \theta_2 + 90°).$$

The student will find the above expressions for the primary and secondary currents can be deduced by analytical processes from the simultaneous equations

$$L\frac{di_1}{dt} + M\frac{di_2}{dt} + Ri_1 = E\sin pt,$$

$$N\frac{di_2}{dt} + M\frac{di_1}{dt} + Si_2 = 0,$$

which equations can be established for two circuits by analogous methods to that by which in § 11 a current equation was arrived at for one circuit, subject to a simple periodic electromotive force.* It is easily seen that if n is very great, or the alternations extremely rapid, then

$$\frac{I_1}{I_2} = \frac{N}{M}.$$

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* *See* Mascart and Joubert's "Electricity," Vol. I., p. 521. (To be obtained at the office of *The Electrician*. In 2 vols.)

If the primary and secondary circuits consist of two equal circuits so interwound that for these circuits $L = M = N$, then for very rapid alternations we see that the secondary current I_2 is equal in magnitude, and exactly opposite in phase, to the primary current I_1, and the magnetic fields due to these currents respectively are also equal and opposite in direction at every instant.

CHAPTER IV.

MUTUAL AND SELF INDUCTION.

§ **1.—Researches of Prof. Joseph Henry.**—An early worker in the field of electro-magnetic discovery opened up by Faraday was Prof. Henry, of Princetown, New Jersey, and his work was fruitful in many new and important investigations. An account of his experiments is to be found in the *Philosophical*

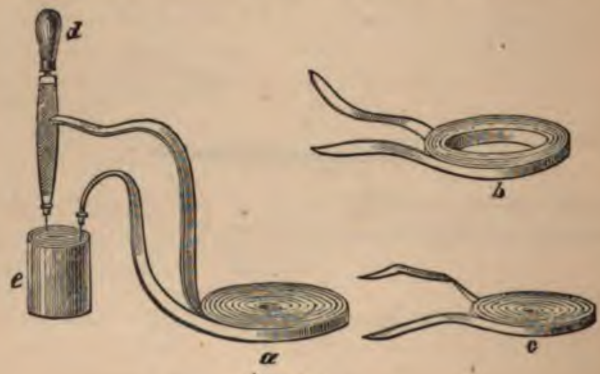

FIG. 1.

Magazine, Vol. XVI., 3rd Ser., 1840. (See also *Transactions* of the American Philosophical Society, 1838, Vol. VI.) As this memoir sets before the reader in a very clear manner the chief phenomena of mutual induction we shall reproduce here the main portions of it.

The principal articles of apparatus used in the experiments consisted of a number of flat coils of copper riband, which were designated by the names *Coil No. 1*, *Coil No. 2*, &c., also several long bobbins of wire, and these, to distinguish them from the ribands, were called *Helix No. 1*, *Helix No. 2*, &c.

Coil No. 1 was formed of thirteen pounds of copper plate one inch and a-half wide and ninety-three feet long; it was well covered with two coatings of silk, and was generally used in the form represented in Fig. 1, which is that of a flat spiral sixteen inches in diameter. It was, however, sometimes formed into a ring of larger diameter, as is shown in Fig. 2.

Coil No. 2 was also formed of copper plate of the same width and thickness as coil No. 1. It was, however, only sixty feet long. Its form is shown at *b* in Fig. 1. The opening at the centre was sufficient to admit helix No. 1. Coils No. 3, 4, 5, 6, were all about sixty feet long, and of copper plate of the same thickness, but of half the width of coil No. 1.

FIG. 2.

Helix No. 1 consisted of sixteen hundred and sixty yards of copper wire $\frac{1}{49}$th of an inch in diameter; No. 2 of nine hundred and ninety yards, and No. 3 of three hundred and fifty yards of the same wire. These helices were wound on bobbins of such size as to fit into each other, thus forming one long helix of three thousand yards, or, by using them separately and in different combinations, seven helices of different lengths. The wire was covered with cotton thread saturated with bees' wax, and between each stratum of spires a coating of silk was interposed.

Helix No. 4, shown at *a*, Fig. 2, was formed of five hundred and forty-six yards of wire $\frac{1}{49}$th of an inch in diameter, the several spires of which were insulated by a coating of cement.

Helix No. 5 consisted of fifteen hundred yards of silvered copper wire, $\frac{1}{125}$th of an inch in diameter, covered with cotton, and is of the form of No. 4.

In addition, a long spool of copper wire covered with cotton, $\frac{1}{16}$th of an inch in diameter and five miles long, was provided. It was wound on a small axis of iron, and formed a solid cylinder of wire eighteen inches long and thirteen in diameter.

For determining the direction of the induced currents a magnetising spiral was used, which consisted of about thirty spires of copper wire in the form of a cylinder, and so small as just to admit a sewing needle into the axis.

Also a small horseshoe is frequently referred to, which was formed of a piece of soft iron about three inches long and $\frac{3}{8}$ths of an inch thick ; each leg was surrounded with about five feet of copper bell wire. This length was so small that only a current of considerable quantity could develop the magnetism of the iron. This instrument was used for indicating the existence of such a current. The battery which was used was a simple copper-zinc cylinder battery, having about $1\frac{3}{4}$ square feet of zinc surface. In some experiments a series of cells was used, but most experiments were performed with one or two cells of the above kind. The manner of interrupting the circuit of the conductor was by scraping one end of the conductor along a rasp, held in contact with the battery terminal.

Provided with this apparatus, a preliminary series of experiments was made on the self-induction of these various coils. The mode of operating was to close the battery circuit by dipping the ends of a coil or helix into two mercury cups, and then to break the circuit by lifting out one end from its mercury cup, the hands being at the same time in contact with the battery terminal and the end of the conductor which is being raised. In this way the *extra current* or electro-magnetic discharge of the coil passed through the operator's body.

When the electromotive force was small, as in the case of a thermopile or a large single cell, and the circuit is the flat riband coil No. 1, ninety-three feet long, it was found to give brilliant snaps at the surface of the mercury when contact was broken, but the shocks were very feeble, and could only be felt in the fingers or through the tongue. The induced current in a short coil, which thus produces deflagration, but not shocks, may, for distinction, be called one of quantity.

When the length of the coil was increased, the battery being the same, the deflagrating power decreased, while the intensity of the shock continually increased. With five riband coils making an aggregate length of three hundred feet and a small battery, the deflagration was less than with coil No. 1, but the shocks are more intense.

There appears to be, however, a limit to this increase of intensity of the shock, and this takes place when the increased resistance or diminished conduction of the lengthened coil begins to counteract the influence of the increasing length of the current. The following experiment illustrates this fact.

A coil of copper wire $\frac{1}{18}$th of an inch in diameter was increased in length by successive additions of about thirty-two feet at a time. After the first two lengths, or sixty-four feet, the brilliancy of the spark began to decline, but the shocks continually increased in intensity until a length of five hundred and seventy feet was obtained, when the shocks also began to decline. This was, then, the proper length to produce the maximum effect with a single battery and a wire of the above diameter. With a battery of sixty cells (Cruickshank's trough), having plates four inches square, scarcely any shock could be obtained when the coil formed a part of the circuit. If the length of the coil was increased, then the inductive effect became very apparent.

When the current from ten cells of the above-mentioned trough was passed through the large spool of copper wire the induced shock was too severe to be taken through the body. Again, when a small battery of twenty-five cells having plates one inch square, which alone would give but a very feeble shock, was used with helix No. 1, an intense shock was received from the induction when the contact was broken. Also a slight shock in this arrangement was given when the contact is formed, but it was very feeble in comparison with the other. The spark, however, with the long wire and compound battery was not as brilliant as with the single battery and short riband coil.

When the shock is produced from a long wire as in the last experiments, the size of the plates of the battery may be very much reduced without a corresponding reduction in the intensity of the shock. A small battery was made, formed of six pieces of copper bell wire, about one inch and a-half long, and an equal number of pieces of zinc of the same size. When the current from this was passed through the five miles of wire of the spool the induced shock was given at once to twenty-six persons joining hands.

With the same spool, and the single battery used in the former experiments, no shock, or at most a very feeble one, could be obtained.

The induced current in these last experiments may be considered as one of *considerable intensity* and *small quantity*.

The energetic action of the flat riband coil in producing self-induction led Prof. Henry to conclude that it would also be the most proper means for the exhibition and study of the phenomena of mutual induction.

§ **2. Mutual Induction.**—Coil No. 1. (*see* Fig. 3) was arranged to receive the current from a small battery of a single cell,

Fig. 3.

and coil No. 2 was placed over it with a plate of glass between to secure perfect insulation. As often as the circuit of No. 1 was interrupted a powerful secondary current was induced in No. 2. When the ends of the secondary were joined to a magnetising spiral the enclosed needle became strongly magnetic. Also when the ends of the second coil were attached to a small water decomposing apparatus, a stream of gas was given off at each pole, and when the secondary current was passed through the wires of the iron horse-shoe magnetism was developed. The shock, however, from the secondary coil was very feeble, and scarcely felt above the fingers. This secondary current had, therefore, the properties of one of moderate intensity but considerable quantity when developed by the current in one flat riband coil acting on another flat riband coil.

Coil No. 1 remaining as before, a longer coil, formed by uniting Nos. 3, 4, and 5, was substituted for No. 2. With this arrangement as a secondary circuit, the magnetising power of the current and the brilliancy of the spark at breaking contact was less than before, but the shocks were more powerful—in other words, the intensity of the induced current was increased, whilst its quantity was decreased.

A compound helix formed by uniting Nos. 1 and 2 helices, and therefore containing two thousand six hundred and fifty yards of wire, was next placed on coil No. 1. The weight of this helix happened to be precisely the same as that of coil No. 2, and hence the different effects of the same quantity of metal (as secondary circuit) in the two forms of a long and short conductor could be compared. With this arrangement the magnetising effects with the apparatus above-mentioned disappeared. The sparks were much smaller and the decomposition less than with the short coil, but the shock was almost too intense to be received with impunity except through the fingers of one hand. The secondary current in this case was one of small quantity but of great intensity.

The following experiment is important in establishing the fact of a limit to the increase of the intensity of the shock as well as the power of decomposition with a wire of given diameter.

Helix No. 5, consisting of a wire $\frac{1}{125}$th of an inch in diameter, was placed on coil No. 2, and its length increased to about seven hundred yards. With this extent of wire neither decomposition nor magnetism could be obtained, but shocks were given of a peculiarly pungent nature. The wire of the helix was further increased to about fifteen hundred yards; the shock was now found to be scarcely perceptible in the fingers.

As a counterpart to the last experiment coil No. 1 was formed into a ring of sufficient internal diameter to admit the great spool of wire, and with the whole length of this (five miles) the shock was found so intense as to be felt at the shoulder when passed only through the forefinger and thumb. Sparks and decomposition were also produced, and needles rendered magnetic. The wire of this spool was $\frac{1}{16}$th of an inch in diameter; and we therefore see from this experiment that by increasing the diameter of the wire its length may be also increased with increased effect.

The previous experiments were repeated, using a battery of sixty cells (Cruickshank's trough). When the current from this was passed through the riband coil No. 1, no indication, or a very feeble one, was given of a secondary current in any of the coils or helices arranged as in the preceding experiments. But when the long helix No. 1 was placed as a primary, instead of coil No. 1, a powerful inductive action was produced on each of the circuits used as before.

First, helices Nos. 2 and 3 were united into one coil and placed within helix No. 1, which still conducted the battery current. With this disposition a secondary current was produced, which gave intense shocks but feeble decomposition, and no magnetism in the soft iron horse-shoe. It was therefore one of intensity, and was produced by a battery current also of intensity. Instead of the helix used in the last experiment for receiving the induction (secondary), one of the coils, No. 3 (copper riband), was now placed on helix No. 1, the battery remaining as before. With this arrangement the induced current gave no shocks, but it magnetised the small horse-shoe, and when the ends of the coil were rubbed together produced bright sparks. It had, therefore, the properties of a current of quantity, and it was produced by the induction of a current from a battery of intensity.*

This experiment was considered of so much importance that it was varied and repeated many times, but always with the same result; and it therefore established the fact *that an intensity current can induce one of quantity*, and by the preceding experiments the converse has also been shown, *that a quantity current can induce one of intensity.*

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* This last experiment is very interesting, as showing that in 1838 Prof. Henry had already realised that which is sometimes called the reverse use of the induction coil. He had employed a current flowing in a fine wire of many turns and moving under a high electromotive force, to induce a current of greater strength in a secondary circuit, consisting of a lesser number of turns of copper riband, and moving under a less electromotive force. In other words, he had constructed *a step-down transformer*. Note Prof. Henry's explicit statement in the following paragraph.

§ 3. **Induction at a Distance.**—In the experiments on mutual induction detailed above, the primary circuit was separated from the secondary only by a pane of glass, but the action was so energetic that an obvious experiment was to investigate the effect of distance on the mutual induction. For this purpose coil No. 1 was formed into a ring of about two feet in diameter (*see* Fig. 4), and helix No. 4 placed as shown. When the helix was at the distance of about sixteen inches from the middle of the plane of the ring, shocks could be perceived through the tongue, and these rapidly increased in intensity as the helix was lowered, and when it reached the plane of the ring they were quite severe. The effect, however, was still greater when

Fig. 4.

the helix was moved from the centre to the inner circumference, as at *c*, but when it was placed without the ring, in contact with the outer circumference at *b*, the shocks were very slight, and when placed within, but with its axis at right angles to that of the ring, not the least effect could be observed. Coil No. 1 remaining as before (the primary) helix No. 1, which was nine inches in diameter, was substituted for the small helix in the last experiment, and with this the effect at a distance was much increased. When coil No. 2 was added to coil No. 1, and the currents from two small batteries sent through, these shocks were distinctly perceptible through the tongue when the distance of the planes of the coil and the three helices united as one was increased to thirty-six inches. The action at a distance was still further increased by coiling the long wire of the large spool into the form of a ring of four feet in diameter, and placing parallel to this another ring formed of the four ribands of coils Nos. 1, 2, 3, 4. When a current from a single cell, having thirty-five feet of zinc surface, was passed through the riband conductor, shocks through the tongue were felt when the rings were separated to a distance of four feet. In another experiment to illustrate induction across a distance Prof. Henry (*Phil. Mag.*, Vol. XVIII., 1841, 3rd Series, p. 492) joined all his coils, so as to form a single conductor of about 400 feet in length, and this was rolled into a ring of five and a-half feet in diameter and suspended vertically against a door. On the other side of the door, and opposite to the coil, was placed a helix formed of upwards of a mile of copper wire one sixteenth of an inch in thickness and wound in a hoop of four feet in diameter. With this arrangement, and with a battery of one hundred and forty-seven square feet of zinc surface divided into eight elements, shocks were perceptible on the tongue when the two conductors were separated by a distance of seven feet, and at a distance of between three and four feet the shocks were quite severe.

In the fifty years which have elapsed since Prof. Henry performed the experiments described above, the progress of knowledge has placed in our hands an appliance vastly more delicate than physiological shock for detecting induction at a distance, viz., the articulating telephone receiver. Aided by this, it has recently been found possible to find indications of the mutual induction between conductors separated by miles instead of feet.

Along the Gray's Inn-road, London, the English Post-office service has a line of iron pipes buried underground carrying many telegraph wires. The United Telephone Company has a line of open wires along the same route over the housetops, situated 80ft. from the underground wires. Considerable disturbances were experienced on the telephone circuit, and even Morse signals were read, which were said to be caused by the continuous and parallel telegraph circuits. A very careful series of experiments,* extending over some period, proved unmis-

* Mr. W. H. Preece on "Induction between Wires and Wires" (*The Electrician*, Vol. XVII., 1886, p. 410; *British Association Report*, Birmingham, 1886).

takably that it was so, and that the well-known pattering disturbances due to induction are experienced at a much greater distance than was anticipated.

Experiments conducted on the Newcastle Town Moor extended the area of the disturbance to a distance of 3,000ft., while effects were detected on parallel lines of telegraph between Durham and Darlington at a distance of 10¼ miles. But the greatest distance experimented upon was between the east and west coast of the Border, when two lines of wire 40 miles apart were affected the one by the other, sounds produced at Newcastle on the Jedburgh line being distinctly heard at Gretna on a parallel line, though no wires connected the two places.

Distinct conversation has been held by telephone through the air without any wire through a distance of one quarter of a mile, and this distance can probably be much exceeded.

Effects are not confined to the air, for submarine cables half a mile apart in the sea disturb each other. It may well be doubted whether the inductive effects above described as taking place over very large distances above mentioned are not complicated by leakage, and that the true inductive effect would be too small to be observed. The question of the limiting distance of detectable induction would require for its settlement the highest attainable perfection in insulating the circuits and special qualifications in the locality in isolation from all other telegraph and telephone circuits.

Practical application of induction across large air spaces has been made in the methods of carrying on telegraphic communication with railway trains when in motion. There are two methods by which this has been accomplished. (1) The magneto-induction method, which was devised by Mr. L. J. Phelps and tried about the year 1885 on a line about 15 miles long between Haarlem River and New Rochelle Junction, in the United States. In the other system (2) the principle involved is that of electrostatic induction, and after having been suggested in a more or less imperfect form by Mr. W. Wiley Smith in 1881 (U. S. Patent No. 247,127) has been worked out in great detail by Messrs. Edison and Gilliland.

In the magneto-induction system a telegraphic car attached to the train carries a great circuit of wire wound on a frame extending the whole length of the car, and so placed that one side of the windings is as near the track as possible, and one side as high above as the height of the car will permit. Between the rails is laid down a fixed insulated conductor, and the fluctuations of a current in this last induce currents in the lower side of the large coil carried on the car. The secondary current so induced is detected by a telephone and by suitable interruptions. A Morse code of audible signals can be transmitted from the fixed conductor to the moving train. The signals are thus made to jump over the air space, and continuous communications can be kept up between a station or stations in connection with the fixed conductor and a person in the moving telegraph car.

Mr. Phelps used a conductor of No. 12 (A.G.) insulated wire, which was placed in a sort of small wooden trough mounted on blocks attached to the sleepers. The car containing the telegraphing instruments carried beneath its floor, and about 7in. above the rail level, a 2in. iron pipe, in which was a rubber tube holding about 90 convolutions of No. 14 (A.G.) copper wire, so connected as to form a continuous circuit about a mile and a-half long, and presenting something like three-quarters of a mile of wire parallel to the primary line wire mounted between the metals. The instruments consisted of a delicate polarised relay as a receiving instrument, which acted as a sounder, and a "buzzer," or rapid current-breaker, for transmitting signals by means of the Morse key, which were received at the station in a telephone. This arrangement was so far a practical success that Mr. Phelps was encouraged to proceed; but meantime it was discovered that the patent above referred to had already been issued, while Edison and Gilliland had also been working on similar lines. In Wiley Smith's specification no mention is made of a "buzzer," which turns out to be an important feature in the invention; but the practical success of the experiments made is due to a combination of the devices of Phelps, Edison, and Gilliland. The latest system is an improvement on that of Phelps, briefly described above, in that it dispenses with the insulated line wire laid between the metals, and uses ordinary telegraph wires strung on what are known as short poles alongside the permanent way. The line wire is, in fact, stretched on poles about 16ft. high and at an average distance of 8ft. from the rails. About 54 miles of the Lehigh

Valley Railroad, U.S., from Perth Junction to Easton, have been so fitted, and on that piece of road experiments were made in 1887. As a rule the roof of the car, usually sheathed with metal, is available for securing the necessary electrical condition, but where a metal covering is absent, all that is necessary is to attach a wire or rod to the roof and another to some portion of the metallic or rolling part of the coach in order to obtain "earth." The instruments are inserted in this circuit, and comprise a 12-cell chromic acid battery (the cells being 2in. wide by 4in. deep), which is closed on an induction coil having a primary of about 3 ohms and a secondary of about 500 ohms, and provided with an ordinary vibrating make-and-break. The messages are sent by means of a Morse key placed in the secondary circuit, this key being of the double-pointed kind with extra contact. The receiving telephone has a resistance of about 1,000 ohms; but Mr. Phelps states that even when wound so as to have a resistance of 10,000 ohms the sound is quite clear, so high is the electromotive force of the induction on the roof. The car-roofs are frequently of metal, usually painted tin plates, sheet zinc, or galvanised iron, and these answer admirably as inductive receivers; but where the roofs are of wood, covered with painted canvas, an iron or brass rod or tube, ½in. in diameter, is carried along under the eaves throughout the length of the train. The metallic roof or the rod is connected by a wire to the secondary of an induction coil, while the primary of the coil is connected to the front contact of the double-pointed key, and through that with the battery. Opposite the core of the coil is the "buzzer," which transmits a series of impulses to the line whenever the key is worked. The extra contact, which is placed on the upper surface of the front contact of the key, closes the secondary circuit, and allows the charges to be sent into the roof, while, when the key is on the back contact, the secondary and primary coils are cut out, and the charge from the roof then passes direct to the key and through the telephone to earth, which, as a rule, is made by connecting wires from the coil and the telephone to one of the axle-boxes. The coil and the key, with suitable connections, are mounted on a board which is large enough to contain a telegraph form besides, and the telephone is attached by flexible connections, and is, when in use, strapped to the operator's head. The battery is put up in a case with a handle, so that the whole apparatus can be carried from one end of a train to the other. The arrangements at the terminal and other stations on the line, so far as induction telegraphy is concerned, are practically identical with those in the railway coach; but in addition they have a duplex Morse equipment, by which ordinary messages can be sent by the dot-and-dash system.

FIG. 5.

§ 4. **Induced Currents of Higher Orders.**—Experiments to be referred to presently led Prof. Henry to the conclusion that the secondary currents, though only of momentary duration, should also be able in their turn to induce others in neighbouring conductors, and it was found on trial that this was the case.

A primary current was passed through coil No. 1, while coil No 2 was placed over it to receive the secondary current, and the ends of this last coil joined to a third coil, No. 3. By this disposition the secondary current passes through No. 3, and since this was at a distance (*see* Fig. 5), and beyond the influence

FIG. 6.

of the primary, its separate induction could be rendered manifest by the effects on helix No. 1, arranged as a secondary circuit to this third coil. When the handles *a b* of the last helix *d* were grasped a powerful shock was received, proving the induction of a *tertiary current* in the last coil. By a similar more extended arrangement (shown in Fig. 6) shocks were received from currents of a *fourth* and *fifth* order; and with a more powerful primary current and additional coils, a still greater number of successive inductions might be obtained.

It was found that with a small battery a shock could be given from the current of the third order to 25 persons joining hands; also shocks perceptible in the arms were obtained from a current of the fifth order.

When the long helix is placed over a secondary current generated in a short coil, and which is one of *quantity*, a tertiary current of *intensity* was obtained capable of producing shocks. When the intensity current of the last experiment was passed through a second helix and another flat riband coil placed over this (*see* Fig. 7), a quantity current is again produced. Therefore, in the case of these currents of higher orders

FIG. 7.

a quantity current can be induced from one of intensity, and the converse.

The arrangement in Fig. 6 shows these different results produced at once. The induction from coil No. 3 to helix No. 1 produces an intensity current, and from helix No. 2 to coil No. 4 a quantity current.

The next stage in Prof. Henry's inquiry had reference to the direction of these induced currents. Bearing in mind that a current on starting in a conductor induces an *inverse* or oppositely directed induced current in a neighbouring secondary circuit, and a *direct* or like directed induced current on stopping, it is evident that analogy points to the fact that each tertiary current must consist, in its simplest form, of two oppositely directed currents, succeeding each other instantaneously. For at the "make" or "break" of the primary the secondary circuit is traversed by a brief secondary current in "opposite" or "like" direction. We shall speak of these as the *inverse* and *direct* secondary currents produced on closing or opening the primary circuit.

Each of these secondary currents rises to a maximum and then sinks to zero again. If there is a tertiary circuit present, then during the rise of the secondary current to its maximum it is developing an *inverse* tertiary, and during its decrease to zero a *direct* tertiary current. Since, as we shall see, the duration of the secondary current is a very small fraction of a second, these two component tertiary currents must succeed each other at an excessively short interval of time. Physiologically their separate effect is, so to speak, lumped into one, and they make themselves felt as a shock. Prof. Henry adopted the method of employing a magnetising spiral containing a sewing needle as a means of detecting the presence of these induced currents of higher order. By inserting such a spiral in the circuit of the successive conductors and noting the direction of the magnetisation of the steel needle he arrived at the conclusion that there exists an alternation in the direction of the currents of the several orders, and that the directions of the several induced currents could be expressed by saying that at the "make" of the primary we get an inverse secondary, a direct tertiary and inverse quarternary current, and so on; or, symbolically:—

Primary current	*started*		*stopped*	
Secondary ,,	*inverse*	−	*direct*	−
Tertiary ,,	*direct*	+	*inverse*	+
Quarternary ,,	*inverse*	−	*direct*	−
	&c.		*&c.*	

The use of a magnetising spiral as a means of determining the direction of an induced current is, however, liable to lead to errors in drawing conclusions as to direction of currents, and the above experiment cannot be regarded as an exhaustive examination of the whole phenomena of induced currents of higher orders. Before entering into a more detailed discussion of the exact nature of the effects which here present themselves, it will be of assistance to gather together the principal observations on the induction of transient electric currents.

§ 5. **Inductive Effects Produced by Transient Electric Currents.**—Since secondary currents, which have, as we shall see, very brief duration, can in turn generate induction currents of higher orders, it was an obvious inference that Leyden jar discharges or the transient currents formed by discharging charged conductors, should in like manner be able to give rise to a family of induced currents in suitably placed circuits. Prof. Henry opened up this field of research, and it has been also diligently cultivated by Marianini, Abria, Matteucci, Reiss, Verdet, and many other physicists. Henry's first experiment was as follows: A hollow glass cylinder (*see* Fig. 8) of about six inches in diameter was prepared with a narrow riband of tinfoil about thirty feet long pasted spirally around the outside, and a riband of the same length pasted on the inside, so that the cor-

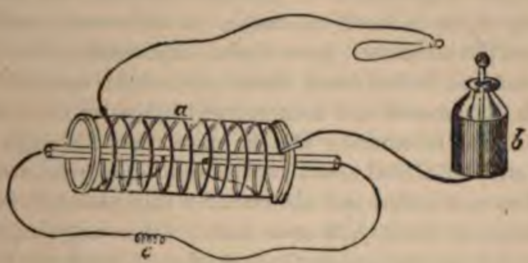

FIG. 8.

responding spires of the two were directly opposite each other. The ends of the inner spiral passed out of the cylinder through a glass tube to prevent direct communication between the two circuits. When the ends of the inner riband were joined by the magnetising spiral containing a sewing needle and a discharge from a half-gallon jar sent through the outer riband, the needle was strongly magnetised in such a manner as to indicate an induced current through the inner riband in the *same direction* as that of the current of the jar. If instead of using the magnetising spiral the ends of the inner riband were brought near together, a small spark was detected at the instant of sending a jar discharge through the outer conductor. Experiments were next made in reference to the production of induced current of different orders by electric discharges. For this purpose a series of glass cylinders with tinfoil spirals pasted on them was prepared and joined up so that the inner spiral of one cylinder was in connection with the outer spiral of another. When a discharge was passed through the outer riband of the first cylinder it produced an induced secondary discharge circulating in the inner spiral of the first and the outer spiral of the second cylinder. This in turn generated a tertiary current, and so forth. Each of these discharges was a brief wave of current, and by the use of the magnetising spiral in each circuit an attempt was made to determine the direction of the discharge. Here, however, an anomaly presented itself. By the use of this magnetising spiral it appeared that the induced discharges were all in the *same direction*. Leyden jar discharges were then passed through the first member of the series of coils and helices used in the experiments on galvanic currents, and here the directions of the induced discharges in the several conductors were found to *alternate*. After various experiments Prof. Henry considered that he had found the solution of this anomaly in the different distances of the inducing and inductive circuits. As an experiment illustrating this he gives the following:—Two narrow strips of tinfoil about twelve feet long were stretched parallel to each other, and separated by thin plates of mica to the distance of about $\frac{1}{50}$th of an inch. When a discharge from a half-gallon jar was passed through one of these an induced current in the *same* direction was obtained from the other. When the ribands were separated to a distance of about $\frac{1}{8}$th of an inch, no induced current, as evidenced by the absence of effect in the magnetising spiral, could be obtained. When the circuits were still further separated the induced current reappeared, but in the *opposite* direction to the primary discharge. The distance at which the induced discharge changes direction appears, according to Prof. Henry, to be dependent on a number of circumstances, such as the capacity and charge of the jar, and on the length and thickness of the wires.

With a battery of eight half-gallon jars and parallel wires of about ten feet long, the change in direction did not take place until the wires were separated by twelve or fifteen inches.

The currents of all the higher orders were found to change sign with a change in the distance between the inducing and inductive circuit.

One interesting experiment was made by Henry to illustrate the induction effect of jar discharges across considerable distances. In this case a primary circuit was formed consisting of an insulated wire eighty feet long. Around this and separated from it by a distance of about twelve feet was another wire one hundred and twenty feet long. When the discharge from thirty large Leyden jars was sent through the primary wire an induced discharge was obtained in the other sufficiently strong to magnetise to saturation a small needle placed in a magnetising spiral interpolated in the secondary circuit. We may, however, remark here, once for all, that all these experiments directed to determine the direction of induced discharges in which the magnetising power of the discharge is made use of for this purpose are difficult to interpret, and too much reliance must not be placed on the conclusions thus drawn. Leaving out of account for the moment all consideration of what are called electric oscillations, to which we shall allude subsequently, we can say that if two discharges are passed through a magnetising spiral, the discharges being oppositely directed and of equal *quantity* but different *durations*, the resulting direction of magnetisation will be dependent upon several conflicting elements. Speaking generally, the intensity of magnetisation is determined by the relative magnitude of the maximum current strength during the discharge, and of two discharges having equal quantity, the one lasting the shortest time would rise to the highest current strength during the period of the discharge, and exercise the greatest magnetising force. Even then it would not be safe to draw too pronounced a conclusion from the direction of magnetisation as to the relative magnitudes of the maxima of two alternate discharge currents rapidly succeeding each other, for, as Abria has pointed out* long ago, the demagnetisation of a steel needle requires a less magnetising force than that necessary to magnetise it in the first instance, and hence the final results are complicated by the relative order of imposition as well as the relative maximum magnitude of the magnetising discharge currents. One fact which has to be borne in mind in attempting to

* Abria, *Ann. de Chem. et de Phys.*, [3] Vol. I., p. 429, 1844.

interpret these results of Prof. Henry is that the magnetising current whose direction we are seeking to determine acts by induction on the mass of the needle or iron in the testing magnetising coil, and generates in its mass induction currents circulating round its surface. Under the head of *Magnetic Screening* in a later section we shall examine the circumstances under which such currents induced in a metallic mass shield to a greater or less extent conducting circuits lying beyond them from inductive effects. Meanwhile we can say that the effect of a very *sudden* discharge in one direction in the magnetising coil is to induce eddy currents in the surface of the needle which shield the inner and deeper portions of the steel from the magnetising action, and the resulting magnetisation is chiefly superficial. If, however, the discharge is prolonged or dragged out whilst retaining the same electric quantity, the shielding action will not be so pronounced, and the magnetisation will penetrate deeper down into the mass of the steel. Accordingly two equal discharges, *i.e.*, discharges of equal quantity, may produce a greater or less magnetic moment in the steel, according as the duration of the same is greater or less, a *very* sudden discharge having much less relative magnetic-moment-producing power than the same quantity more dragged out. We may in general also say that the magnetising power of a discharge current is determined by the value of the maximum current strength during the discharge, and hence, of two equal quantity discharges, the one which lasts the shorter time, and which has, therefore, the greatest maximum value, will, if the discharges are approximately equal in duration, produce the greatest magnetising effect.

The tertiary currents, produced by ordinary galvanic currents, and the secondary currents, produced by Leyden jar discharges, consist, as we have seen, in their simplest form of a double discharge or flow, one part *inverse*, or oppositely directed to its inducing parent current, and the succeeding part *direct*, or similarly directed, the two component currents of the total discharge having equal quantity but different durations. In general the first portion, or the inverse current, is that which has the greatest maximum value and the shortest duration, the second half, or the direct current, being more dragged out in time; and, for a reason to be stated further on, the approximation of the induced and inducing circuits exaggerates this difference, or increases the maximum value of the inverse current at the expense of its duration. The explanation which may be offered, then, of the phenomena of the magnetisation of steel by tertiary currents, or by the secondary currents due to Leyden jar discharges, is as follows:—When the induced and inducing circuits are not very near to each other, and when the inducing current reaches its maximum not very suddenly, the two halves of the induction current are not very different in duration, but the first or inverse current has, of the two, a rather greater maximum and less duration. There results then a magnetisation in the needle, which, on the whole, is in the direction produced by the inverse current, and the inference from the direction of magnetisation is that the induced and inducing currents are in the opposite direction. If, however, the inducing current reaches its maximum value very suddenly, as it does if the circuits are very close, then the first half, or the inverse induced current, is so brief in its duration that the magnetisation of the needle due to it is very superficial. On the other hand, the magnetisation due to the rather more prolonged direct current is more diffused through the needle, and the resultant magnetisation found on testing the needle is that apparently due to the *direct* current, and the inference from the resulting magnetism of the needle would be that the induced and inducing currents are in the *same* direction. By some such explanation as the above we may reconcile these anomalous results of Prof. Henry with known facts, but it is evident, since the resulting magnetisation of the needle is an effect determined by the relative maximum values of the two portions of the total induced current, and by their duration, as well as by their order of superposition, that considerable caution is necessary in attempting to interpret the results of experiments made with a magnetising helix. Henry was followed in the same field of investigation by Abria, Marianini, Reiss, and Matteucci. Matteucci endeavoured to determine the direction of the induced discharges by employing a process founded upon the experiment of the pierced card, in which the hole made by the spark on a piece of paper or a card is always nearer to the negative electrode. By means of this process, combined with the employment of the galvanometer, Matteucci considered that the inductive discharges are determined by the following law :—If the inducing and induced circuit are both closed, the induced discharge is in the *opposite* direction to the inducing discharge. If, however, the induced circuit is interrupted at any point so that there is a spark, the induced discharge is in the *same* direction as the inducing. Abandoning these methods above described, M. Verdet* employed another, which depends upon the *polarisation of electrodes* in dilute sulphuric acid.

From more recent knowledge we may state the facts with regard to the action of alternate currents upon a dilute sulphuric acid voltameter as follows† :—

If a current of electricity consisting of alternate short fluxes of current of opposite sign is passed through a voltameter having platinum electrodes, and if these electrodes are large, there is no visible decomposition, but if the electrodes be reduced in size below a certain limit visible decomposition begins. For every current there is a certain size of electrode, below which gas is not visibly evolved, and for every given size of electrode there is a current below which gas is not apparently liberated. When the conditions are suitable for the liberation of gas the gases collected at both electrodes have the same composition. If the quantities of electricity passing in each alternate and oppositely directed flux are equal, then the electrodes are not sensibly polarised. If, however, the quantities are not equal, then there is, on the whole, a greater flow of current in one direction than the other, and the electrodes exhibit the state known as polarisation, and yield a reverse current when connected with the galvanometer. Verdet, in his experiments, made use of flat spirals, the wires of which were insulated from each other with great care by silk and a layer of gum-lac varnish. The primary spiral was made of copper wire $\frac{3}{8}$ths of an inch in diameter and 92 feet in length, forming 24 spirals. The secondary circuit consisted of three spirals of wire $\frac{1}{50}$th of an inch in diameter and 157 feet in length, making 95 turns. The inducing discharge was supplied from a Leyden jar battery of 9 large jars. The induced discharge was sent through a voltameter

* Verdet, *Ann. de Chem. et de Phys.*, [3] Vol. XXIX., p. 501, 1850.

† *See* a Paper by MM. Maneuvrier and J. Chappuis, abstracted in *The Electrician*, June 29, 1888, Vol. XXI., p. 237.

having small platinum electrodes, and which could be connected with a delicate galvanometer for detecting polarisation of the electrodes immediately after the discharge. M. Verdet's experiments led him to recognise that when the induced circuit is continuous, and not interrupted anywhere except by the insertion of the voltameter, no traces of polarisation are obtained except by very powerful discharges. This indicates that the induced discharge consists of a double current of two oppositely directed and equal quantities of electricity. In the case of very powerful discharges there was a slight galvanometric deflection, indicating a preponderating secondary discharge in the *same* direction as the primary. If the induced or secondary circuit is interrupted at one point, so that the discharge has to pass as a spark at that place, then very perceptible polarisation of the electrodes presents itself, and the direction of this is such as to indicate a predominant induced current passing in the *same* direction as the primary.

To sum up. It follows from all the numerous researches on induced discharges that this is a very complex phenomenon, and is influenced by a large number of conditional circumstances, and also by the very mode employed for determining it. It may be, however, taken as proved that an induced discharge, produced either as a secondary discharge by a transitory primary, such as the discharge from a Leyden jar, or a tertiary current produced by induction by a secondary current of very brief duration, is in its simplest form a wave of electric current, consisting of two short fluxes or currents in opposite directions, and succeeding each other immediately. This Poggendorff* holds to be shown by the action of such tertiary or higher order currents on a galvanometer. If these currents are led through a galvanometer of which the arrangement is such that the magnetic axis of the needle is accurately at right angles to the direction of the magnetic axis of the coil, then no deflection of the needle is observed, or at most a very slight one. If, however, the needle makes an angle with the plane of the coils, then these induction currents cause a deviation of the needle. This effect (*der doppelsinnigen Ablenkung*) arises from the fact that the magnetism of the needle is not rigid, and that the

* Poggendorff, *Pogg. Ann.*, Bd. XLV., p. 353, 1838.

alternate twisting couples to which the needle is subjected are not equal, by reason of the fact that one of the halves of the complete induced current—say the direct half—increases the magnetic moment of the needle, and hence increases slightly the deflecting couple in the direction tending to increase the deviation of the needle; the other half—say the inverse part of the induced current—tends to reduce the moment of the needle, and hence to subject it to a smaller reverse couple. Hence it follows that, if discharges of equal quantity and opposite sign succeed each other through a galvanometer when the needle is accurately in the plane of the coils, little or no deviation is observed; but if the coils are turned so that the needle makes

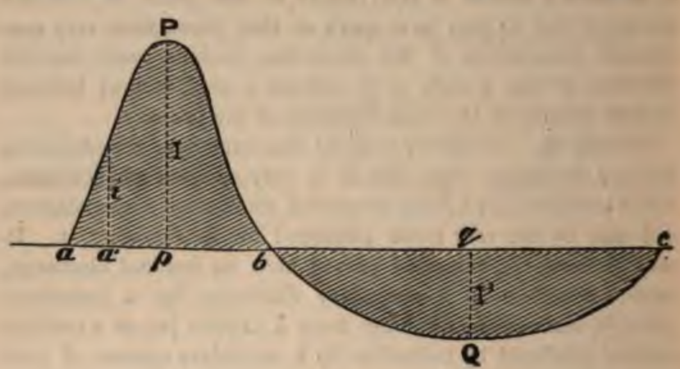

Fig. 9.

an angle with them, then these alternate currents will affect the needle and increase the angle of deflection.

This behaviour towards a galvanometer, and the action on a voltameter of liberating mixed gases of equal composition at each pole prove that each induced current of the third and higher order consists of two oppositely directed discharges, produced by the operation of two successive electromotive impulses of opposite sign and very brief duration acting upon the circuit. The quantities of these discharges are equal; but the durations are different, and hence the maximum value of the current strength during the opposite discharges may be very different.

This may be illustrated graphically thus :—

Let the curve a P b Q c (Fig. 9) be a current curve representing two waves of current of opposite sign succeeding each other. Let the horizontal line $a\,c$ be a time line, and vertical ordinates represent instantaneous current strengths. Then the areas (shaded) will represent the *quantity* in each discharge. Let these shaded areas be equal, then the diagram represents two discharges of equal quantity succeeding each other in opposite directions, but having very different maximum current strengths I and I′. The duration of the first discharge is represented by $a\,b$, and that of the second by $b\,c$. This diagram represents, then, what we find in the simplest case of tertiary current. If the instantaneous value of the current at any time is called i, then the whole quantity of the discharge will be represented by the shaded area and by the integral $\int i\,d\,t$ between proper limits.

We may classify the effects of induced discharges or currents in the following way:—

(1) Those effects dependent upon $\int i\,d\,t$, or upon the whole quantity of the discharge. These are the *galvanometric* and the *electro-chemical* effects. If a discharge is passed through a galvanometer, the duration of which is very small compared with the time of free oscillation of the needle, the galvanometer needle experiences a "throw" such that the sine of half the angle of deflection is proportional to the whole quantity of the discharge. Also in a voltameter, by Faraday's law, the whole quantity of the electrolyte broken up is proportional to the quantity of electricity which has pass through it.

(2) Those effects dependent upon $\int i^2\,d\,t$, or upon the average of the square of the strength of the current at every instant during the discharge. These are the *heating* and the *electro-dynamic* effects. By Joule's law at every instant the rate of dissipation of energy is proportional to the square of the current strength, and hence the whole heat generated by the discharge is proportional to the integral above. Similarly, if the discharge passes through a circuit, part of which is movable and can react upon a fixed part, so that attraction or repulsion may take place between them, the force is dependent at any instant

on the square of the current strength, and hence the whole effect or average force upon the same integral.

(3) We have, lastly, effects dependent chiefly upon the maximum ordinate I, or upon the rate of change of the current—that is, upon the steepness of the slope of the current curve. These are the *physiological*, *telephonic*, *luminous*, and *magnetic*.

The physiological effect of a discharge in giving a shock appears to depend in great part upon the suddenness with which the maximum current strength is reached. Of two discharges which reached equal maxima, that which arrived at it in the shortest time would be the most effective in producing shocks. The value of the maximum current strength is also important. Two induced currents of equal quantity but different duration cause a greater shock in proportion to their lesser duration. The telephone in this respect resembles the animal body. It is affected more by the rate of change of the current strength than by the absolute current strength at any instant.

The *magnetic* effect depends, as has been shown by Lord Rayleigh,* upon the maximum current strength during the discharge, or upon the initial current strength, in those cases in which the current dies gradually away. In the two Papers referred to below it is shown by direct experiment that, since the time required for the permanent magnetisation of steel is small compared with the duration of induced currents generally, the amount of acquired magnetism depends essentially on the initial or maximum current strength during a transitory current, without regard to the time for which it lasts. It is, then, not difficult to understand that the effort to settle by experiment with a magnetising coil the direction of induced discharges may lead to very conflicting results, and, in any case, it is hardly competent to do more than indicate the direction in which the maximum current flow takes place during the discharge.

The *spark* effects are also included in this category. The air or other dielectric is broken down when the difference of potentials between the two discharging points reaches a certain

* See *Phil. Mag.*, Ser. 4, Vol. XXXVIII., 1869, p. 8: The Hon. J. W. Strutt, "On some Electromagnetic Phenomena." Also *Phil. Mag.* Ser. 4, Vol. XXXIX., 1870, p. 431.

magnitude, and in the case of a varying electric pressure the question whether a spark will pass or not is evidently determined by the maximum magnitude of that quantity.*

It is evident from the above considerations that the complete analysis of the effects and phenomena of induced currents of the higher orders, and of those of secondary currents due to discharges from condensers, requires a knowledge of the *form* of the current curve in each case. We proceed to consider the problem of the theory of induced currents in some of its simpler aspects.

§ 6. **Elementary Theory of the Induction Coil.**—Aiming rather at the elucidation of principles than very copious treatment, we shall consider the problem of the induction coil in its simplest form. Let there be two bobbins of wire in suitable positions for producing mutual induction and without iron cores. Let the constant inductance of the first or primary be denoted by L and its ohmic resistance R, and the similar quantities for the second or secondary coil be N and S, and let M be the coefficient of mutual inductance.†

Let there be a source of constant electromotive force, E, which can be applied or withdrawn from the primary circuit. We denote by x the strength of the current in the primary at any time t after closing the primary circuit by applying the battery to it. Also we denote by y the current in the secondary circuit at any time reckoned from the same zero.

If, then, at any instant the currents are x and y, the following state of things exists in the circuits.

The electromotive force E is the impressed force on the primary circuit.

The effective electromotive force producing the current x is Rx, the counter-electromotive force of self-induction is $-L\frac{dx}{dt}$, the counter-electromotive force of mutual induction due to the current y at that instant in the secondary circuit is $-M\frac{dy}{dt}$,

* *See* Bertin, "Notes on Electrodynamic Induction," *Ann. de Chimie*, 4th Ser., Vol. XXII., April, 1871, p. 486.

† Continental writers often call L and N the potentials of the bobbins on themselves, and M the potential of one bobbin on the other.

and hence the relation which at any instant holds good between these quantities is

$$L\frac{dx}{dt}+M\frac{dy}{dt}+Rx=E,$$

which is an expression of the fact that the external electromotive force balances the internal electromotive forces, and the effective electromotive force driving the current.

Similarly, for the secondary circuit we have an induced electromotive force due to the induction of the primary on the secondary equal to $M\frac{dx}{dt}$ and a counter-electromotive force of self-induction $N\frac{dy}{dt}$; hence

$$N\frac{dy}{dt}+M\frac{dx}{dt}+Sy=0,$$

since there is no external impressed electromotive force. The complete solution of the problem of finding the currents x and y at any instant is obtained by the solution of these simultaneous differential equations—

$$L\frac{dx}{dt}+M\frac{dy}{dt}+Rx=E,$$

$$N\frac{dy}{dt}+M\frac{dx}{dt}+Sy=0.$$

As our object is not specially to deal with difficulties of analysis we shall simplify the problem by supposing that the two circuits are similar in every respect. This makes $R=S$ and $L=N$, and the equations become

$$L\frac{dx}{dt}+M\frac{dy}{dt}+Rx=E \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$L\frac{dy}{dt}+M\frac{dx}{dt}+Ry=0 \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

Bearing in mind that L is in ordinary parlance the "number of lines of force" which are linked with the primary circuit due to unit current in its own circuit, and that M signifies the number of lines of force which are common to both, or linked in with both circuits, when unit current flows in each, we see that M can never be greater than L, but that under all circumstances we must have

$$M \le L,$$

also

$$M \le N;$$

hence

$$M^2 \le L\,N,$$

or $L\,N - M^2$ always a positive quantity, and the maximum value which the co-efficient of mutual inductance M can have is $\sqrt{LN}$, or the square root of the product of the self-inductances.

In order to separate the differentials in (i.) and (ii.) we differentiate each equation with respect to t, and obtain—

$$L\frac{d^2 x}{d t^2} + M\frac{d^2 y}{d t^2} + R\frac{d x}{d t} = 0 \quad . \quad . \quad . \quad \text{(iii.)}$$

$$L\frac{d^2 y}{d t^2} + M\frac{d^2 x}{d t^2} + R\frac{d y}{d t} = 0 \quad . \quad . \quad . \quad \text{(iv.)}$$

Multiply (iii.) by L, (iv.) by $-M$, and (i.) by R, and then adding the three equations together we obtain—

$$\frac{d^2 x}{d t^2} + \frac{2\,L\,R}{L^2 - M^2}\frac{d x}{d t} + \frac{R^2}{L^2 - M^2}x = \frac{E\,R}{L^2 - M^2} \quad . \quad . \quad \text{(A)}$$

and a similar elimination gives us

$$\frac{d^2 y}{d t^2} + \frac{2\,L\,R}{L^2 - M^2}\frac{d y}{d t} + \frac{R^2}{L^2 - M^2}y = 0 \quad . \quad . \quad . \quad . \quad \text{(B)}$$

We have now separated the differentials in x and y, and the solution of these equations depends, as is well known,* upon the solution of an auxiliary quadratic equation—

$$m^2 + \frac{2\,R\,L}{L^2 - M^2}m + \frac{R^2}{L^2 - M^2} = 0,$$

the solution of which is—

$$m = -\frac{R}{L+M}, \text{ or } -\frac{R}{L-M}.$$

* *See* Boole's "Differential Equations," p. 192, 2nd Edition.

Hence the general solution of (A) and (B) is—

$$x = Ae^{-\frac{Rt}{L+M}} + Be^{-\frac{Rt}{L-M}} + \frac{E}{R}, \quad . \quad . \quad . \quad (5)$$

and
$$y = A'e^{-\frac{Rt}{L+M}} + B'e^{-\frac{Rt}{L-M}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where A, B, A′, B′ are constants of integration to be determined from the circumstances of the flow. To do this, however, a preliminary discussion is necessary. Let us suppose that the primary current is fully established, and has a steady value I, and hence that M I lines of force penetrate through the secondary circuit. This quantity is then the electro-magnetic momentum of the secondary circuit, because when the current in the primary is steady there is no current in the secondary circuit.

Let us now suppose that the primary circuit is broken, and that the circumstances of the "break" are such that all these M I lines of force are removed at a uniform rate in a small time, δt, from the secondary circuit.

During this time δt an electromotive force will operate upon the secondary circuit equal in magnitude to $-\frac{MI}{\delta t}$, or to the rate of decrease of the included lines of force. We have seen in Chap. III. that when an electromotive force E acts on a circuit of inductance L and resistance R that the current i at any time t after the commencement of the application of the electromotive force is given by the equation

$$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

Now, in the case considered the inductance and resistance of the secondary circuit are L and R, and the impressed electromotive force applied during a time δt is $\frac{MI}{\delta t}$. Hence, at any time *during* the interval of time δt the value of the secondary current is given by the equation

$$i = \frac{MI}{\delta t R}\left(1 - e^{-\frac{R}{L}t}\right).$$

This gives us the value of the inverse induced current at any time during the breaking of the primary Expand the above expression by the exponential theorem and it becomes

$$i=\frac{M}{L}I-\frac{M}{L^2}\frac{R\delta t}{1\cdot 2}+\frac{M}{L^3}\frac{R^2\delta t^2}{1\cdot 2\cdot 3}-\text{etc.}$$

At the instant when the removal of lines of force or the cessation of the induction through the secondary takes place the impressed electromotive force ceases and the secondary current begins to die away. If we suppose the "break" of the primary to be very sudden, δt becomes practically zero, and we have

$$i=\frac{M}{L}I;$$

that is to say, the secondary current starts with a value equal to $\frac{M}{L}$ of that of the steady primary.

The state of things in the secondary circuit immediately *after* the break of the primary is, then, this: the electromotive impulse due to stoppage of the primary has generated a current of initial value $\frac{M}{L}I$ in the secondary, but there is no impressed electromotive force in the secondary circuit. If at any instant after the break the current in the secondary circuit is i, the law of decay of this current is expressed by the equation

$$L\frac{di}{dt}+Ri=0.$$

The solution of this is

$$i=Ce^{-\frac{R}{L}t}$$

and the constant C is found from the condition that when $t=0$ $i=\frac{M}{L}I$. Hence we have

$$i=\frac{M}{L}Ie^{-\frac{R}{L}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This gives us the value of the direct or "break"-induced current in the secondary at any instant after the break of the primary. Graphically, this may be represented by a curve, such as that in Fig. 10. During the time O T in which the primary is being broken the induced electromotive force is creating an induced current, the rising strength of which is represented by the rise O P. The time occupied by the break δt is O T. As O T is diminished in value, the magnitude of the maximum ordinate P T approximates to $\frac{M}{L}I$, and this is the initial value of the inverse secondary current when the break is very sudden. After the break the current decays away along a path represented by P Q, and becomes zero only after an infinite time.

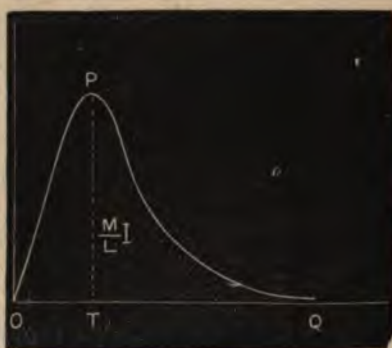

FIG. 10.

The whole quantity of the induced current is obtained by integrating equation (7) with respect to the time from zero to infinity, thus:

$$\int_0^\infty i\,dt = \int_0^\infty \frac{M}{L} I\, e^{-\frac{R}{L}t}\,dt = \frac{M I}{R}.$$

We see, then, that both the maximum value and whole quantity of the direct secondary current are proportional to the coefficient of mutual induction and to the strength of the primary current, and, moreover, that the whole quantity of electricity set in motion in a secondary circuit of total resistance R by suddenly removing from it M I lines of force is equal to the quotient of number of lines removed by the total resistance of the secondary circuit.

If the induced current is sent through a galvanometer the indications are proportional to the magnitude of $\frac{\text{M I}}{\text{R}}$. If, however, the induced current is employed to magnetise steel needles, the magnetisation acquired is dependent upon the magnitude of $\frac{\text{M I}}{\text{L}}$, and is therefore greater in proportion as the coefficient of self-induction of the secondary circuit is less. Lord Rayleigh has pointed this out,* and shown by experiment that within certain limits the magnetising effect of the break-induced current on steel needles is *greater* the smaller the number of turns of which the secondary consists, the opposite being, of course, true of the galvanometer. The galvanometer takes account of the induced current as a whole; whilst the magnetising power depends mainly on the magnitude of the current at the first moment of its formation, without regard to the time which it takes to subside.

Returning to the equations (5) and (6) we can now find the constants of integration, counting the time from the instant of "make" of the primary. It is obvious that when $t=0$, $y=0$ and $x=0$, and that the whole quantity of the make-induced current, or $\int_0^\infty y\,dt$, must be equal to the whole quantity of the break-induced current, which we have seen is equal to $\frac{\text{M I}}{\text{R}}$.

In (6) put $t=0$, $y=0$; we get

$$\text{A}'+\text{B}'=0, \qquad \text{or } \text{B}'=-\text{A}'.$$

Hence,

$$y=\text{A}'\left(e^{-\frac{\text{R}t}{\text{L}+\text{M}}}-e^{-\frac{\text{R}t}{\text{L}-\text{M}}}\right)$$

and

$$\int_0^\infty y\,dt=-\frac{2\,\text{A}'\text{M}}{\text{R}}.$$

Hence the whole quantity of the "make"-induced current is $-\frac{2\,\text{A}'\text{M}}{\text{R}}$, and this must be equal to $\frac{\text{M I}}{\text{R}}$, which is the whole quantity of the "break" current. Hence $\text{A}'=-\frac{\text{I}}{2}$.

* *Phil. Mag.*, Ser. 4, Vol. XXXIX., 1870, p. 429.

Therefore we get for the instantaneous strength of the "make" secondary current

$$y = -\frac{I}{2}\left(e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right) \quad . \quad . \quad . \quad (8)$$

Again in (5) put $t=0, \quad x=0,$

and we get $A+B+I=0,$

or $B=-(I+A);$

and by substitution in (5)

$$x = A\,e^{-\frac{Rt}{L+M}} - (A+I)\,e^{-\frac{R}{L-M}} + I.$$

From this equation we can find the value of A by substituting the value of $\frac{dy}{dt}$ derived from equation (8), and $\frac{dx}{dt}$ derived from the above in the original differential equation (i.), and we find $A=-\frac{I}{2}$. Hence we arrive at the equation for the value of the primary current at any instant, and it is

$$x = I - \frac{I}{2}\left(e^{-\frac{Rt}{L+M}} + e^{-\frac{Rt}{L-M}}\right) \quad . \quad . \quad . \quad (9)$$

This gives the law according to which the primary current grows up in its circuit. If $M=0$, that is, if there is no secondary circuit; then

$$x = I\left(1 - e^{-\frac{Rt}{L}}\right)$$

which is the ordinary law of current growth. If $M=L$, which is the greatest possible value of M, then

$$x = I\left(1 - \frac{1}{2}\,e^{-\frac{Rt}{2L}}\right).$$

Hence it is obvious that the presence of the secondary circuit hastens the rise of the primary current and operates on it to reduce its inductance.

On making the primary we get a "make" or inverse secondary current according to the law of growth expressed by the equation

$$y = -\frac{I}{2}\left(e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right),$$

and we see that under the circumstances assumed the "make" secondary starts from an initial value zero, rises up to a maximum, and then decays away again. To find the time of reaching maximum, equate $\frac{dy}{dt}$ to zero, and we find

$$t' = \frac{L^2 - M^2}{2\,R\,M} \log\left(\frac{L+M}{L-M}\right),$$

and this function *increases* as M *decreases*. So that the more nearly M is equal to L the *sooner* does the secondary reach its maximum. It is not difficult to show that when $M = L$ the above value for t' becomes zero, and when $M = 0$ $t' = \frac{L}{R}$.

If, then, we trace a series of curves (Fig. 11) representing the values of y, or the make-induced current at each instant for various and increasing values of $\frac{L}{M}$ as the coils are moved further apart, we find a series of curves with decreasing maxima, but the maxima happening later as M decreases.

Lastly, on breaking the primary current we have a break-induced current in the same direction as the primary, which, at any instant after the "break," is decaying away according to the law

$$z = \frac{M}{L} I e^{-\frac{Rt}{L}}.$$

If the break was absolutely instantaneous, the induced current would start with a finite value equal to $\frac{M}{L}$ of that of the primary, but as no form of break entirely eliminates sparking, the rise of the direct secondary current is a gradual one. Also we have another element of disturbance which enters into the case. The self-induction of the primary creates direct electromotive force in its own circuit at the instant when the induction through the primary due to its own current vanishes. When the primary is broken either at a mercury cup or at a platinum point the fusion and volatilisation of metal which takes place keeps open for a little time a conductive path through which flows the *extra current* due to the self-induction of the primary. This extra current may have at a particular instant a higher value than that of the strength of the steady current, so that for a short instant the primary current may be actually increased on beginning to break the circuit before it commences to decay.

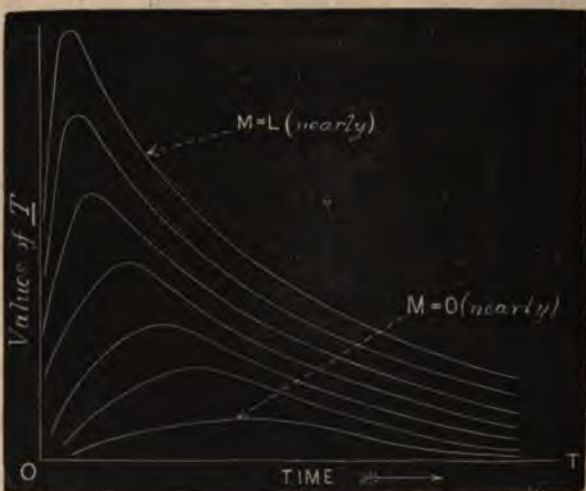

FIG. 11.

Curves representing roughly the current value of the make-induced current for different and increasing values of M.

This direct extra current in the primary will have its effect in introducing a very short inverse-induced current, which will precede the main direct-induced current due to the decay of the primary current. In any event it will introduce an electrical oscillation tending to render the growth of the direct secondary current a gradual matter. It is an interesting case to examine the relative maximum values and duration of the two induced currents under an assumption very nearly realised when the primary and secondary are wound together on the same bobbin, viz., when $M = L$. In this case the values of y and z become

$$y = \frac{I}{2} e^{-\frac{Rt}{2L}}$$

$$z = I e^{-\frac{Rt}{L}}.$$

The maximum of the direct currents ("break") is I, and that of the inverse (or "make") is $\frac{I}{2}$. If we wish to know at the end of what times t and t' the strengths of the two induced currents y and z are reduced to $\frac{1}{m}$ of that of the primary we obtain by substitution of $\frac{1}{m}$ for y and z in the two above equations the following :—

$$\frac{1}{m} = e^{-\frac{Rt}{L}} \quad \text{for the direct-induced current,}$$

and $$\frac{1}{m} = \frac{1}{2} e^{-\frac{Rt'}{2L}} \quad \text{for the inverse-induced current,}$$

and therefore $$\frac{t'}{t} = 2\left(1 - \frac{\log 2}{\log m}\right).$$

We see that t' is always greater than t, and that in proportion as m increases, t' tends towards a limit $2t$, or the inverse current has a duration about double that of the direct secondary. We shall now see how this theory is confirmed by experiment.

§ 7. **Comparison of Theory and Experiment.**—Masson and Breguet carried out a series of experiments on induced currents. The principal part of their apparatus was a commutator keyed on a revolving shaft, which enabled them to separate the direct and inverse-induced currents. Two brass wheels were keyed on one shaft, but insulated from it, and the wheels had depressions cut in their periphery which were filled up with ivory. These wheels could be shifted relatively to each other, and were insulated from each other and from the shaft (*see* Fig. 12). Two springs pressed against the edge of the wheels, and two against the hub of the wheel. The whole arrangement served as a means to break and make one circuit, and at the same time to control a second circuit so that it was broken at the time when the first was made, and made at the time when the first was broken, or *vice versâ*. One of these wheels was inserted in the circuit of a primary coil and battery, and the other in the circuit of a secondary coil and galvanometer. On rotating the wheel at a certain fixed speed the series of "break" and "make" induced currents are separated out; all one set are stopped out and all the other are sent through the galvanometer. In this way it was shown that the quantities of the induced currents were equal, but very different

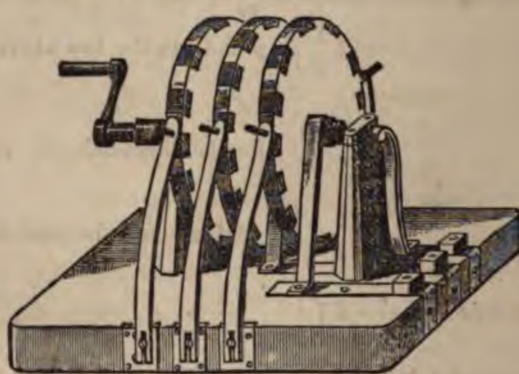

FIG. 12.

in maximum magnitude, and hence in duration, the break-induced currents being greatly superior in making sparks.

Lenz* wound a spiral of wire on the soft iron armature of a magnet and connected the ends of the wire to a ballistic galvanometer. He detached the armature suddenly, and observed the throw of the galvanometer. If θ denotes the angle of deflection and x the number of windings, he found that the product $\frac{1}{x} \sin \frac{\theta}{2}$ was a constant quantity, which shows that, *cæteris paribus*, the quantity of electricity set in motion was in proportion to the number of lines of induction with-

* Lenz, *Poggendorff's Annalen*, Bd. XXXI., p. 385, 1835.

drawn from the circuit. He also established experimentally, in confirmation of Faraday, that the electromotive force of induction was independent of the width, thickness or material of the wire windings,* and by other experimentalists also the fact has been established that the electromotive force is independent of everything except the form of the conductor and the nature of the change it experiences in relation to the magnetic induction through it. Felici † carried out an extensive series of experiments on induction, using a form of induction balance.

In this apparatus a secondary circuit, consisting of two coils, is arranged in series with a galvanometer. These coils are so

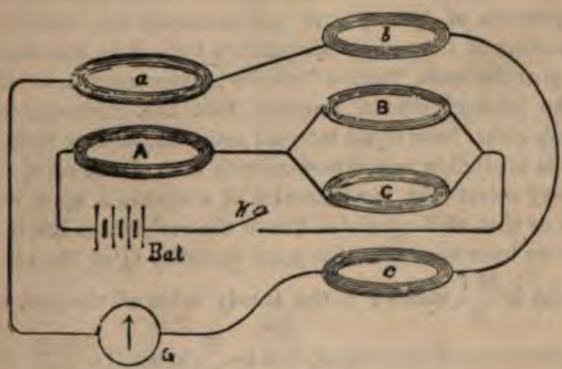

Fig. 13.

far apart as not to influence one another. In contiguity to each secondary coil is a primary coil, and the primaries are wound in opposite directions. The primaries are in circuit with a battery and a key. The circuits can be so arranged, by adjusting the distances of the coils, that the induction of the primaries on their respective secondaries balance each other, and the galvanometer indicates no current, however strong may be the primary current. If three pairs of coils (*see* Fig. 13) are

* *See* Faraday, "Exp. Researches," Ser. II., § 193, *et seq.*; also for electrolytic circuits, *see* L. Hermann, *Pogg. Ann.*, 1871, p. 586.

† Felici, *Nuovo Cimento*, Vol. IX., 1859, p. 345, also *Ann. de Chimie* [3], Vol. XXXIV., 1852, p. 64.

thus taken and balanced, two and two, so that the induction of A on *a* is equal to that of B on *b* and C on *c*, then, if we connect the primary A in series with B and C in parallel, so that the current divides between them in the ratio of their resistances, and connect the secondaries with a galvanometer, all in series, so that the current in *a* is opposed to that in *b* and in *c*, then no induced current is detected when the battery circuit is made and broken. This proves that the quantity of the induction current is proportional to the strength of the primary current.

If a primary and secondary coil are taken in fixed positions and the "throw" of a galvanometer observed when a definite steady electromotive force E is applied to the primary, then if the position of battery and galvanometer are reversed, the application of the same electromotive force E to the secondary will give the same "throw" on the galvanometer now attached to the primary circuit, provided that the galvanometer and battery either have equal internal resistance or that their resistance is negligible in comparison with that of the coils. Hence we may assert that the induction of a circuit A upon B is the same as that of B upon A. For, if the resistances are R and S, then we have seen that the total quantity Q of the secondary current is $\frac{MI}{S}$, where I is the steady value of the primary and M is the mutual inductance; but $I = \frac{E}{R}$, hence $Q = \frac{ME}{SR}$. If, then, the positions of battery and galvanometer are reversed, we get a quantity of induced current equal to $\frac{ME}{RS}$, which is the same as before. For any two coils it is possible to find a number of relative positions in which the interruption of a current in one produces no induced current in the other. In such cases the coils are said to be *conjugate* to each other. It is manifest that when in these positions the lines of induction produced by one coil do not pass through the other. It is possible to use one coil in this way to explore the field of another.

Let P be a primary coil and S be a small flat secondary coil, both being shown in section in Fig. 14. Then, if S is placed in a position conjugate to P, it will be found possible to move the coil S along a certain line A B C, maintaining the flat face of the coil always tangent to that line and so that in all these positions P and S are conjugate. It is evident that such a line is a line of induction of the coil P.

When one coil is in a conjugate position to the another, as far as regards inductive action they may be considered to be at an infinite distance apart. It follows, therefore, that if a coil is moved suddenly from a conjugate position to one not conjugate in the field of a primary traversed by a steady current, and then the primary current stopped at the instant of arriving at the second position, a galvanometer in the second circuit will have its needle jerked from one position of rest to another

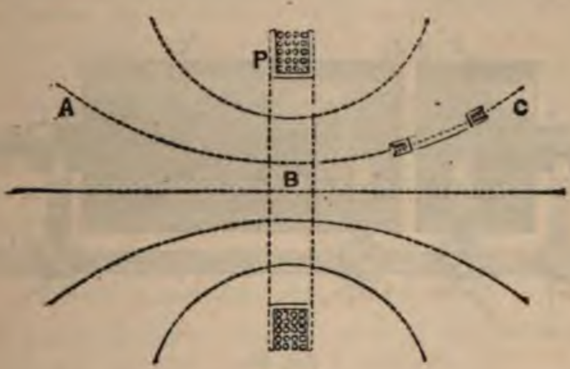

FIG. 14.

of rest, because the interruption of the current takes out of the circuit of the second coil just as many lines of induction due to the first coil as the motion from one position to the other put in. A series of well-devised experiments on the conjugate positions of two coils has been carried out by Mr. W. Grant.*

An elaborate investigation into the duration of induced currents was made by Blaserna.†

* See *Proc.* Physical Soc., London, Vol. III., p. 121; also *Proc.* Physical Soc. London, Vol. IV., p. 361.

† Blaserna, "Sul sviluppo e la durata delle Correnti d'induzione," *Giornale di Science Naturali*, Vol. VI. (Palermo, 1870).

A commutator was constructed which consisted of two insulating cylinders keyed on one shaft and having on part of their surface brass coverings cut into steps (*see* Fig. 15). These cylinders were capable of being set in any relative position to each other on the shaft. The shaft could be revolved at a high rate of speed, and its velocity ascertained by a siren plate attached to the axis. This siren plate consisted of a disc pierced with holes against which was directed a jet of air. From the pitch of the musical note given out, when ascertained by comparison with standard tuning forks, the speed could be determined. Two springs pressed against the hubs of these cylinders and two against the surfaces of these cylinders, and a current entering by the hub was conducted to the brass coating

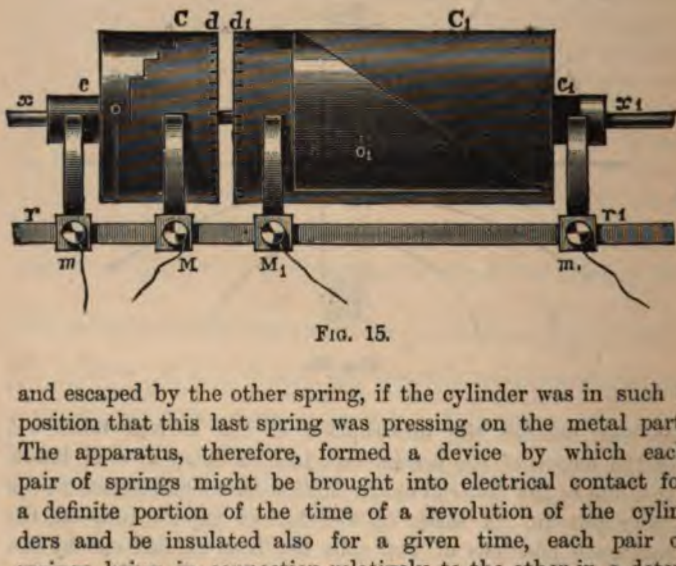

FIG. 15.

and escaped by the other spring, if the cylinder was in such a position that this last spring was pressing on the metal part. The apparatus, therefore, formed a device by which each pair of springs might be brought into electrical contact for a definite portion of the time of a revolution of the cylinders and be insulated also for a given time, each pair of springs being in connection relatively to the other in a determined manner for a determined time. In the circuit of the one cylinder and pair of springs *m* M was placed a battery primary coil, and tangent galvanometer, and in the circuit of the other pair a secondary coil and sensitive galvanometer. This being prepared, the primary coil P and the secondary S were placed a given distance apart. On revolving the commutator it periodically interrupts the primary current, the time during which the primary current is kept on depending upon the position of the spring M on its cylinder. The other cylinder can be so set as to collect either the direct or inverse secondary currents, and send them in series through the sensitive galvanometer, the time during which this secondary circuit is closed being capable of regulation by the adjustment of the spring M_1. In his experiments Blaserna first investigated the duration of each of the induced currents. The interrupters were so arranged relatively to one another that whilst the primary circuit was made and broken, the secondary circuit was not closed until a small time after "making" the primary, and then broken again before the primary was broken. By adjusting the secondary interrupter a position could be found in which the galvanometer just showed no current. The interval between the closing of the primary and the opening of the secondary was then the interval occupied by the secondary current, and this was the duration of the "make"-induced current. Blaserna found that the "make" secondary (inverse) lasts a longer time than the "break" current (direct). For the coils used the times were—

Inverse secondary lasts ·000485 second.

Direct secondary lasts ·000275 second.

He next proceeded to obtain the curve of each current, and to determine the time of arrival at a maximum.

The secondary interrupter was so set that the secondary circuit was closed just before the primary, and opened after at a certain definite interval of time. The galvanometer thus received a current which was made up of repeated doses of the whole quantity of the induced current up to a certain fraction. Knowing the speed of the commutator and the coefficient of the galvanometer, the value of the whole quantity of the induced current, extending over a certain fraction of its whole duration, was known; and from those observations, repeated at regular progressive intervals during the whole period of the current, the value of the ordinates of the current curve can be obtained. For if the curve (Fig. 16) A P P′ B (upper figure) represents the variation of current during a time A B, so that $PX = y$ represents the current strength at a time X and P′ X′ represents the current strength after a very small interval of time, $X X' = d t$. Then the area $P P' X' X = y\, d t$ represents the quantity of electricity which has passed in the time $X X'$. Call this $d Q$.

Hence $d Q = y\, d t$, or $$y = \frac{d Q}{d t}.$$

Suppose another curve $A' P' R$ (lower curve) is drawn on an equal abscissa $A' B'$, such that its ordinate at every point represents the whole area of the upper curve up to the corresponding point—that is to say, the lower curve is a curve such that its

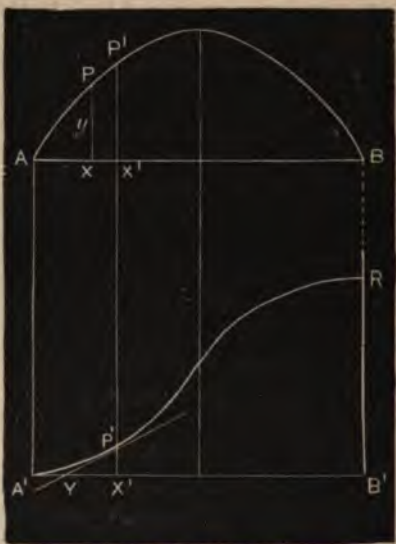

FIG. 16.

ordinate $P' X'$ is proportional to the area $A P X$ of the upper curve, $A X$ (upper curve) being equal to $A' X'$ (lower curve), when the time interval $d t$ becomes very small. It is easily seen that if the area $A P X$ (upper curve) is called Q, and the ordinate $P X$ is called y, that the tangent of the angle $P' Y X'$ (lower curve) which the geometrical tangent drawn at P' makes with the axis $A' B'$, and which is represented by $\frac{d Q}{d t}$, is proportional to the ordinate $P X$. Hence the upper curve is a derived curve of the lower, and if we are given a curve like the lower curve, the ordinates of which represent the whole quantity of electricity which has from a given epoch flowed past a point, we can, by drawing a curve whose ordinates represent the *slope* of the first curve, obtain a second curve, which is a curve of current. In this way it is possible to describe the current curve, and to determine its form and position of maximum.

Blaserna found that the greater the distance apart of the primary and secondary—in other words, the less the mutual inductance—the less was the maximum value of the secondary current, and the greater the delay in the appearance of that maximum. This is in accordance with theory. In the case of the "break," or direct secondary current, he found the delay in establishing the maximum not so great, and the maximum ordinate was greater, though the total duration of the current less. He established by direct experiment the equality of the quantity of the two induced currents. When the coils were very near together the induced current at starting established itself by a series of *electrical oscillations*.

By the help of the same apparatus Blaserna investigated the rise of a current in a coil when the same is placed suddenly in connection with a constant source of electromotive force. For the "make" extra current only one of the revolving interrupters was used, and the circuit was completed by the means of a battery, galvanometer, and coil. When the commutator was revolved it first started the current and then after an interval cut it off again, and the effect on the galvanometer is due to the sum of all these small quantities of electricity so cut off and integrated whilst the current is in process of increasing. As the duration of the time of contact was increased the galvanometer deflection increased (speed of revolution remaining constant), but when the time of contact was long enough to fully establish the current, then increase of speed of rotation did not increase the galvanometer deflection. By this apparatus the fact was established that the primary current established itself in its coil by a series of oscillations.

Similarly, on breaking the circuit the course of the current was investigated. For this purpose one revolving interrupter, I, was inserted in the circuit of a battery, B, and coil, C, and from the ends of the coil (*see* Fig. 17) other wires were brought and led through the galvanometer G, and other interrupter I′, arranged as a shunt on the coil. The break in the battery circuit at *p* was so arranged that each time the current was fully established before being broken again. The break in the galvanometer or shunt circuit was so arranged relatively to the other that the shunt circuit was closed a little before the battery circuit was broken, and then opened at a definite interval afterwards. In this way there was a little flow of current through the galvanometer due to the steady current, but this could be estimated and allowed for. On plotting out a current curve from the quantity curve it was found that the current decayed away on interrupting the circuit by a series of

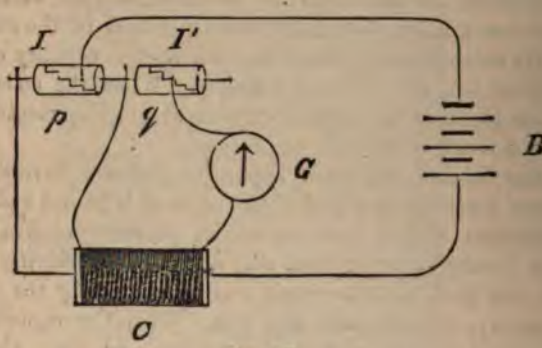

FIG. 17.

oscillations which followed each other much quicker than those on the establishment of it, and the whole duration of the extra current at "break," or the time of falling from steady current to practical zero, was less than the time required to fully establish the current. It was found that the first oscillation, on beginning to interrupt the steady current, had a much greater amplitude than any of those on starting the current.

The duration of an oscillation was perhaps three or four ten-thousandths of a second, and about 50 to 100 oscillations probably happened before the current became steady; hence the whole duration of the variable period, or of the extra current, was about two to three-hundredths of a second. Very roughly, the nature of the oscillatory character of the current at the make and break might graphically be represented by the diagram in Fig. 18.*

Blaserna drew from his observations the deduction that there is an interval of delay in the starting of the secondary currents, and that a small but measurable time elapses between the instant of making or breaking the primary circuit and the beginning of the secondary current. From this he made a calculation as to the velocity of electromagnetic induction, and he also stated that the interposition of dielectric substances

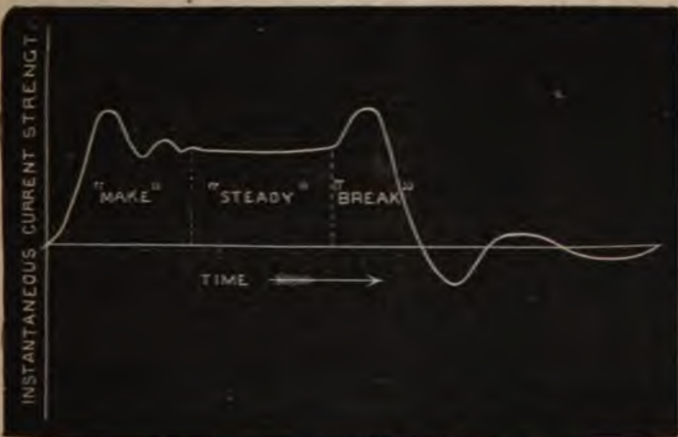

FIG. 18.

such as glass or shellac between the coils reduced the so calculated velocity.

Bernstein (Pogg. *Ann.*, Bd. CXLII., p. 72, 1871) repeated these observations of Blaserna, but did not confirm these last results. He found that the first oscillation always began at the instant of breaking or making the primary circuit, and he found no effect produced by the interposition of dielectric media.

* In *The Electrician* for June 1, 1888, a curve is given by Mr. F. Higgins, showing the rise of current in the magnets of type-printing telegraphs, and the oscillatory character of the current at starting is well marked. Mr. Higgins's curve gives the results of actual observations.

Helmholtz has carefully examined these results of Blaserna and criticised them.* He remarks that Blaserna used for his coils flat spirals of wire with many turns, and also he used the current from several Bunsen cells to create the primary current. Not only do the spirals act like a condenser, giving the whole apparatus a sensible electrostatic capacity, but the use of a battery of high electromotive force causes a considerable spark at the break, which spark has a very sensible and rather irregular duration. Also in Blaserna's experiments, the two circuits were placed at various distances apart. If a current is started in a primary coil the effect of the induced current created in the secondary by its re-action on the primary is to hasten the rise of the primary current, and at the break to accelerate its decay. As the secondary circuit is moved further off this effect is less marked. Hence, the rise and fall of the primary is more gradual and the arrival of the secondary current at its maximum value is more delayed. From this results, then, an apparent retardation of the time of the arrival of the maximum of the induced current.

Helmholtz conducted a series of experiments by means of his pendulum chronoscope. A heavy iron pendulum, P (*see* Fig. 19), the lower end of which carried two plates of agate, could be made to execute one swing and then be caught by a detent. These plates of agate in the course of the swing were caused to strike against and tip over two little levers l, l'. One of these levers was fixed, and the other could be moved forward so as to make the blows successive. One was made to break the circuit of a primary coil, *Pr*, when tipped over, and the other by its movement separated a connection between a condenser and the ends of a secondary coil, *Sec*, attached to it.

These being arranged, the fall of the pendulum executed these two "breaks" successively, separated by an interval of time capable of being calculated from the known motion of the pendulum. The two circuits were placed 170 centimetres apart. The primary consisted of 12 turns of thick wire, and the secondary of 560 turns of fine wire. The current was sent from one Daniell cell. The two ends of the secondary were connected to the two plates of the condenser, and when the pendulum fell it

* Helmholtz, "On the Velocity of the Propagation of Electrodynamic Effects," *Phil. Mag.*, Ser. 4, Vol. XLII., 1871, p. 232.

broke the primary current and started in the secondary circuit an oscillatory current reverberating to and fro in the secondary wire, the condenser acting as a resonator. At a definite interval after rupture of the primary the condenser was separated and examined by a Thomson electrometer. The charge in the condenser showed the *phase* of the electrical oscillation existing at the instant of such separation. In one case Helmholtz observed 35 oscillations in $\frac{1}{80}$th of a second. In order to discover if any

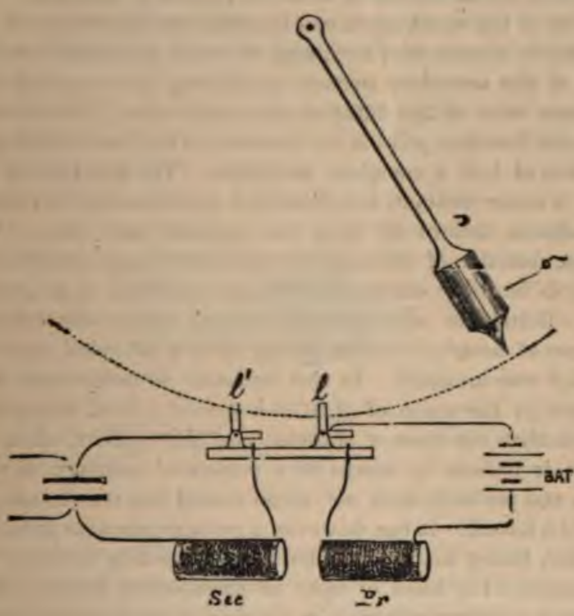

Fig. 19.

retardation took place with increased distance of the coil, it was necessary to fix attention upon some phase in the oscillations. The successive zero points of the current were very sharply defined, and suitable for this purpose. Helmholtz found that alteration of the distance between the primary and secondary coils made no perceptible difference in the position of the zero points, and that, as far as the apparatus he was using could detect, the velocity of the electro-magnetic impulse must be greater than 195 miles per second. He pointed out in this paper that the commencement of the secondary current is not a sharply marked thing. The spark which takes place at break of the primary lasts an appreciable time, and all this time the primary is dying gradually, and the induced current therefore is increasing. The period of duration of the break spark may be something like $\frac{1}{15000}$th to $\frac{1}{45000}$th of a second, and is, therefore, a large fraction of the duration of a single electrical oscillation, which amounted to about $\frac{1}{2811}$th of a second. The duration of the break spark can be found by observation of the time which elapses from beginning of break up to the first zero point of the secondary current oscillations, as compared with the mean value of the duration of an oscillation. The interval up to the first zero point is the duration of the break spark *plus* the time of half a complete oscillation. The duration of the spark is never constant, and depends a good deal on the amount of platinum thrown off from the contacts each time. The average duration of the spark in Helmholtz's experiments was found to be about one-tenth of the whole period of an oscillation. Helmholtz also noticed in some earlier observations evidence of electrical oscillations set up in a flat spiral, one end of which was insulated. In this case some 45 oscillations were detected in the space of $\frac{1}{80}$th of a second. Prof. Henry also noticed that the time of subsidence of the current, when the circuit is broken by means of a surface of mercury, is very small, and probably does not much exceed the ten-thousandth part of a second. It has, however, a quite appreciable duration, for Prof. Henry found that the spark at ending presents the appearance of a band of light of considerable length, when viewed in a mirror revolving at the rate of six hundred revolutions per second.

Bernstein, with the aid of a contact break somewhat different from that used by Blaserna, also examined the duration of the oscillations set up in a secondary coil. He found that the duration of the first oscillation at breaking primary was longer than that of the subsequent ones. The mean duration when using a single Grove cell in the primary circuit was ·0005 second, and when using a Daniell cell only ·0001 second. We shall return later to consider more recent researches on these electrical oscillations.

§ 8. Magnetic Screening and the Action of Metallic Masses in Induction Coils.—If a primary and secondary coil are separated by a metallic sheet, Henry found* that a notable decrease took place in the intensity of the *shock* taken with this arrangement. A thick metal plate was found more effective than a thin one in thus preventing the inductive effect of the primary upon the secondary coil. If a radial slit was cut in a circular metallic plate the annulling effect was altogether stopped. If the two edges of the gap (*see* Fig. 20) were furnished with wires leading to a magnetising spiral, Henry found he could in this way make evident the existence in the plate of a current induced by the

Fig. 20.

action of the primary. A flat coil of insulated wire was substituted for the metal plate, and it was found that the screening action of this coil was only sensible when the two ends were joined so as to complete the circuit. This action, by which the induction of a primary coil on a secondary is prevented by the interposition of a metallic plate cylinder or closed circuit of insulated wire, is called *magnetic screening*. The elementary explanation of this effect is not difficult to arrive at. Suppose a small conducting circuit of resistance R to be placed in a magnetic field so that it is traversed normally by N lines of magnetic induction. Let the constant coefficient of self-induction be L. If, then, in any small time $d t$ a variation of the

* *Phil. Mag.*, Vol. XVI., 1840, p. 257.

lines of induction traversing this circuit takes place, the impressed electromotive force on that circuit will be represented by $-\frac{dN}{dt}$, and if at that instant the current in the circuit is i, by the principles laid down in the last chapter, the current equation will be

$$L\frac{di}{dt}+Ri=-\frac{dN}{dt}$$

or

$$\frac{d}{dt}\left(Li+N\right)+Ri=0.$$

Suppose the conductivity of this circuit to be perfect, and R therefore zero, we have, by integration of the above equation, the result

$$Li+N=\text{const.};$$

in other words, the lines of induction Li, added to the circuit by the induced current generated in it, are *opposite* in direction to those whose variation is producing it, and together with them make up a constant number. Hence, if the variation of N is such as to take lines of induction out of the circuit, the operation of the current thereby induced is to add or increase them in the circuit at an equal rate. If we suppose our circuit to be a perfectly conducting metal plate, and just behind this metal plate there is another small closed circuit, then any variation of lines of induction passing through this plate will not take effect in producing any induction current in the small circuit, because the operation of the current induced in the plate nullifies, as far as the small circuit is concerned, any variation of the external field. It is clear that these conclusions would apply to any surface of finite extent which possessed perfect conductivity; the induced currents which any variation of the external field would produce in this surface would always be such that the induction through each portion would be kept constant—in other words, that the perpendicular component of the magnetic induction at each point on the surface would retain a fixed value. It follows that a closed surface of zero resistance is a complete screen for all points in the interior against the effects of variation of the field on the other side of the surface; these effects reduce to the production of surface currents, which keep the field in the interior constant or at zero.

Faraday describes ("Exp. Researches," Vol. I., § 1720 *et seq.*) an experiment which at first sight seems to disprove the fact of magnetic screening. He placed a flat copper wire spiral, which was in connection with a battery and key, between two other flat spirals which were respectively connected with the two coils of a differential galvanometer. The coils were so joined up that the inductive effect of a break and make of the battery circuit produced no movement of the galvanometer needle because it was subjected to two equal and opposite impulses from the two coils. When an exact balance was obtained a flat plate of copper, nearly three-quarters of an inch thick, was interposed between the primary spiral and one of the secondaries. The galvanometer needle was not, however, any more affected than if the copper was away. To understand this we must bear in mind that the break or make of the primary current produces in the copper a secondary current, but as the effect of the primary coil on the secondary coil on that side is balanced by the other one we may regard the secondary coil next the copper plate as free to receive any inductive effect it can from the eddy current induced in the copper block. This secondary current induced in the copper generates a *tertiary* current in the secondary spiral, and this tertiary current consists, as we have seen, of a double short flux of electricity equal in quantity and opposite in sign. The galvanometer is then traversed by two small equal quantities of electricity in opposite directions, and as this does not sensibly affect a not very sensitive galvanometer no movement of the needle is seen. If, however, instead of the differential galvanometer Faraday had used a differential telephone he would have found distinct evidence of a screening action. Again, suppose that, instead of a simple make or break, Faraday had employed a steadily periodic or alternate current in the primary, this would have set up a steady periodic secondary current of equal frequency in the copper plate, and this again would have set up in the secondary coil on that side a steadily periodic tertiary current of equal period, and this might have been detected by the use of a differential electro-dynamometer or a soft iron needle galvanometer.

Henry found that a sheet of tinfoil afforded a very small amount of screening for shock, but a thick sheet of copper a very considerable one in the case of induction by battery currents, and in the case of induction by Leyden jar discharges the same phenomenon was apparent. In the case of an iron screen there is an additional effect, due to the fact that the iron, by its small magnetic resistance, conducts away the lines of induction somewhat through its mass, and prevents them from extending to the space on the other side. In this case also a considerable thickness of metal is necessary to bring about the effect of annulment. In the case of Sir W. Thomson's ironclad galvanometer a very thick shell of iron is necessary to annul all permanent field in the interior space. When we are limited to the use, as we are in practice, of materials whose conductivity is far from being perfect, it is found that a thin screen of metal hardly affords any sensible protection from inductive effect. In other words, the field on the other side of the screen is very far from constant. This has been well demonstrated in certain investigations by Prof. D. E. Hughes in carrying on some highly valuable experimental researches into the means of preventing induction upon lateral telegraph wires.* It has many times been proposed to annul mutual induction between telegraph and telephone wires by covering them over with thin metal covering, which covering is kept "to earth." It is now known, and well exemplified in Prof. Hughes's experiments, that this shielding affords no protection when the covering is not very thick and when the rate of change of the currents is not very rapid. A gutta-percha wire was enclosed in ten coverings of tinfoil, and such arrangement was not found to afford protection to induction, as detected by a telephonic wire stretched alongside. Even when twenty coatings of thin charcoal iron were put round the wire, not only was there found to be a very sensible permanent field outside the iron, but changes of field were made manifest also. It is not to be taken that these experiments invalidate the fact of magnetic screening, but only that the low conductivity of the envelopes used is ineffective at the speed of current change employed to render visible the effect

* *See* a Paper by Prof. Hughes "On Lateral Induction in Telegraph Wires," read before the Society of Telegraph Engineers, March 12, 1879.

of magnetic screening. It is different, however, if the inductive effects are being produced by a *very rapid rate* of change of field. For suppose that a small circuit as before is placed in a uniform field, and traversed by q lines of induction due to this external field. Suppose q varies according to a simple periodic law, so that $q = Q \cos p t$, where $p = 2 \pi n$, n being the frequency of the alternations. Then we have

$$-\frac{d q}{d t} = Q p \sin p t ;$$

but $-\frac{d q}{d t}$ is the value of the impressed electromotive force in the circuit, and if we call the current at any instant i, then, by the principles in Chap. III., we have

$$i = \frac{Q p}{\sqrt{R^2 + p^2 L^2}} \sin (p t - \theta),$$

in which R is the resistance and L the inductance of the circuit, and

$$\theta = \tan^{-1} \frac{L p}{R}.$$

Suppose that R is very small compared with L p, which is the case when n or the frequency of alternation is made very great, then R vanishes compared with Lp, and if we call i' the value towards which i approximates in this case, we have

$$i' = -\frac{Q}{L} \cos p t,$$

and $$\frac{d i'}{d t} = \frac{Q}{L} p \sin p t,$$

or $$L \frac{d i'}{d t} = Q p \sin p t = -\frac{d q}{d t}.$$

Hence $$L \frac{d i'}{d t} = -\frac{d q}{d t},$$

or $$L i' + q = \text{constant}.$$

Hence the field due to the current in the circuit, together with the external field, is a constant quantity, and we get the condition of perfect shielding. We may sum up the foregoing by saying that if a screen of absolutely no electrical resistance is interposed between a primary and secondary coil, it effects a perfect magnetic screening, whatever may be its thickness. If, on the other hand, the screen has a finite conductivity, then the screening will be very imperfect, unless a very great thickness of material is used, and the above will be true when the change of field or the change of primary current is a simple "make" and "break" or a slowly periodic change. When, however, the change of current in the primary is *very* rapidly periodic, then the screening effects of even imperfect conductors will make themselves felt, and a comparatively thin screen of metal will effect a nearly perfect shielding for induction. This theory is strikingly confirmed by some very beautiful experiments of Mr. Willoughby Smith, which are described in the

FIG. 21.

Journal of the Society of Telegraph-Engineers (November 8, 1883, Vol. XII., p. 458),* and entitled "Experiments on Volta-Electric Induction." Mr. Willoughby Smith's apparatus consisted of two flat coils A and B (*see* Fig. 21), placed a certain distance apart. One of these was a primary coil connected with a battery, and the other was connected with a sensitive galvanometer. In the circuit of both were current reversers, which reversed the galvanometer and battery alternately, and hence made the opposite induced currents both affect the galvanometer in the same direction. This being arranged, the commutator was started so as to reverse the currents very slowly, and a sheet of copper interposed between the spirals. Under these circumstances the interposition of the copper produced but little

* See also *The Electrician*, November 17, 1883, p. 18.

effect. If, however, the commutator was driven at a very rapid rate the copper plate caused a marked diminution in the galvanometric deflection, and this diminution was greater in proportion as the speed was greater. For the original Paper a curve is given (Fig. 22) which shows the decrease in the galvanometer deflection, expressed as a percentage of the original undiminished deflection, corresponding to various speeds of reversal. It will be seen that the *less* the conductivity of the metal the *greater* must be the speed in order that the magnetic screening may approach perfection. Iron, of course, occupies an exceptional position. It cuts off, even at

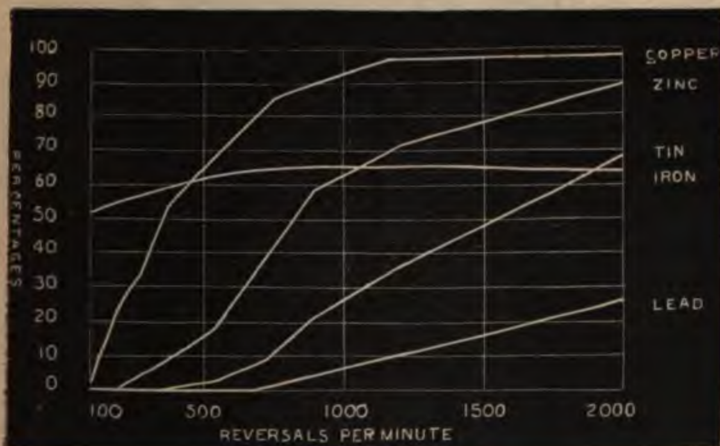

FIG. 22.

very low speed reversals, a large portion of the field, not by a true screening action, but by conducting away the lines of magnetic force and preventing their access to the secondary coil. It will be seen that at any given speed the order in which the metals reduce the deflection is the order of their electric conductivity, and that as far as the diagram goes the lines all (except iron) slope upward, indicating that at very high speeds the screening of even the worst conductors will approach perfection. It would no doubt be found that if the telephone were used as a detector the magnetic screening of a copper plate or thin tinfoil sheet would become very manifest for high notes when not in any way marked or distinguishable for notes or sounds of low frequency of vibration.*

As far back as 1840 Dove had made experiments† on the effect of the introduction of cores of various materials into the

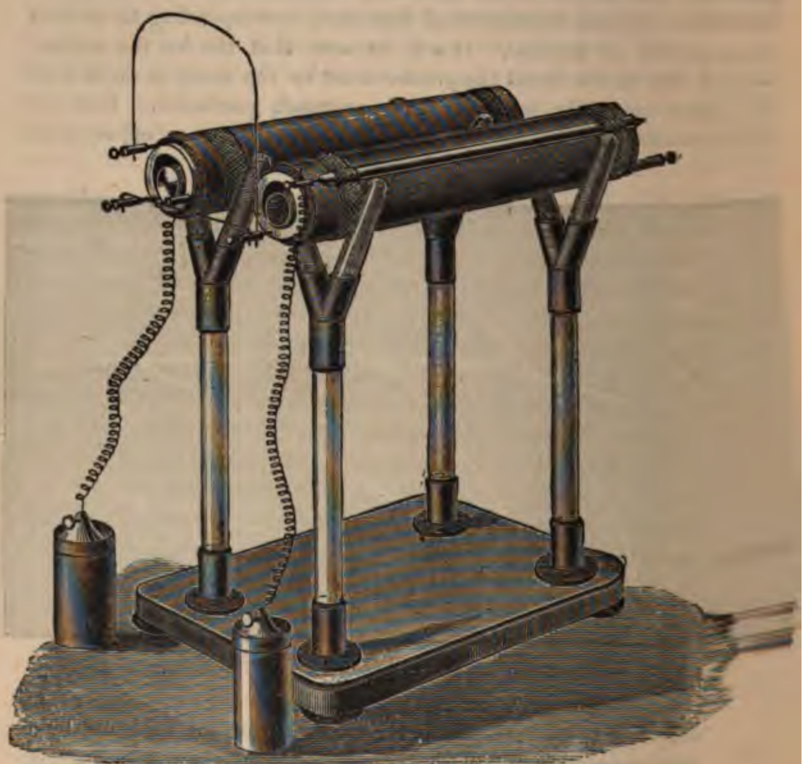

FIG. 23.

primary circuit of an induction coil. His apparatus consisted of two similar primary bobbins wound on tubes of non-metallic

* The above explanation of the cause of the difference between the screening of the different metals is not that given by the distinguished investigator, but it is the explanation which to the author seems most in accordance with known principles.

† Dove, Poggendorff's *Annalen*, Vol. XLIX., 1840.

telephone emits a steady rattle or hum. If a massive copper rod is introduced into the primary bobbin as a core, the telephonic rattle is more or less suppressed; if a core of soft iron wire is introduced the noise is increased; if a core of solid iron or steel is used the noise may be increased, but not so much as when the divided iron is used. The explanation of the exalting effect of the soft iron wire is simple. The presence of the iron reduces the "magnetic resistance" of the circuit of magnetic induction due to the primary current. More lines of induction therefore flow through the secondary circuit, and hence the strength of the secondary current is increased, and the mean rate of change of induction through it is also increased. The diminishing effect of the copper core is explicable by the light of the knowledge that in such a conducting core the primary current generates induced currents, and these in their turn re-act upon the secondary circuit, inducing in it a tertiary current. The direction of the currents induced by the primary in the solid core and in the secondary circuit are the same. The direction, however, of the first half of the tertiary current developed in the secondary by the current in the copper core is opposite to the direction of the current developed in the secondary by the action of the primary. Hence it results that the current in the secondary circuit is more or less wiped out by the opposing inductions due to the primary circuit and the currents induced in the copper core. Otherwise the operation might be regarded thus:—Suppose the primary circuit to be traversed by a periodic current creating a simple periodic flux of induction through the copper core. As we have seen under the head of magnetic screening, this variation of induction would induce currents in the copper core which would themselves generate a flux of induction, which would, if the conductivity of the core were perfect, or the rapidity of change of induction infinite, be exactly equal and opposite at each instant to the flux of induction producing those currents.

If the conductivity is not quite perfect, or the rate of variation not very great, yet nevertheless the direction of the field of magnetic force inside the copper, due to the currents induced in its mass, will more or less oppose the field of force at every instant which is by its fluctuations generating those currents. If the thick white line 1 1 1 in Fig. 24 represents the sinusoidal or simple periodic change of induction or magnetic field in the interior of the copper, due to the primary helix, and if the dotted line 2 2 represents roughly the changing field due to the eddy currents generated in the core, which are *nearly* 180° behind the primary in phase, the integral or sum of both superimposed fields represented by 3 3 at any instant is less than the original one due to the primary alone at the corresponding instant. Also the *mean rate of change* of the resultant field is less, and the secondary circuit experiences at every instant a less inductive electromotive force. The same reasoning which we have employed in the case of magnetic shielding applies here, and the differences in the reducing effect of cores of various metals would be found to be less at high speeds of alternation than at low. In some small induction coils used for medical purposes the strength of the secondary current is

FIG. 24.

graduated by drawing in or out of the primary coil a copper tube which slips over the bundle of fine iron wires used as a core. The *rationale* of the action of this copper tube in so operating is in a rough general way to be found in the principles laid down above.

When Prof. Henry obtained possession of the "Experimental Researches" of Faraday, as detailed in the fourteenth series of his "Experimental Researches," he was exercised in his mind to reconcile the results obtained by Faraday on the interposition of metallic screens between inducing and induced circuits with his own. Faraday had found that when the galvanometer was used as a current finder "it makes not the least difference" whether the space between the primary and secondary coils was air, sulphur, shellac, or such conducting bodies as copper and other non-magnetic metals. On the other hand,

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Prof. Henry found that a shock from a secondary coil which would paralyse the arms was so much reduced by the interposition of a metallic plate as hardly to be sensible on the tongue. Here was evidently something to be explained, and in a long memoir (*Phil. Mag.*, Series 3, Vol. XVIII., 1841, p. 492; also *Transactions of the American Philosophical Society*, Vol. VIII., 1840) Prof. Henry examined this and other matters. He first verified Faraday's experience by attaching the ends of a secondary coil to a galvanometer and bringing up suddenly towards it a permanent magnet, or a coil traversed by a steady current. The swing of the galvanometer was found to be quite unaffected in extent by the interposition of a plate of copper. Again, in place of the copper plate, a closed metallic conductor (an endless coil) was employed, but whether the circuit of this coil was open or closed it made not the slightest difference on the galvanometer deflection.

Forty feet of copper wire, covered with silk, were wound on a short cylinder of stiff paper, and into this was inserted a hollow cylinder of sheet copper, and into this again a rod of soft iron. When the latter was rendered magnetic, by suddenly bringing in contact with its two ends the different poles of two magnets, a current was generated in the wire, but the strength of this current, as measured in the galvanometer, was the same whether the copper cylinder was present or was removed. Prof. Henry then noticed that there was one element of difference between the indications of a galvanometer and that of the magnetising spiral. If the two secondary currents at "break" and "make" of a primary were sent through a magnetising spiral and through a galvanometer, the arrangement might be such that the induced current at "make" of the primary was unable to give any sensible magnetisation to the steel needle enclosed in the spiral, but at "break" was able to magnetise it to saturation. Nevertheless, in both cases the "throw" of the galvanometer was the same. Similarly with the degree of shock felt, the galvanometer indications being alike for the inverse and direct induced current; yet that induced current gave the greatest shock which was able to produce the greatest magnetisation. The explanation of these facts became clear as soon as it was seen that the deflections of the galvanometer depended upon the whole quantity of the discharge, and must necessarily be alike for the inverse and for the direct current, but that the magnetising effect and the physiological shock depended upon the maximum value of the instantaneous discharge current, and might therefore be very different for the two induced currents. It was then evident that any actions by which this maximum value of an induced current was decreased, whilst its duration was increased and total quantity left unaltered, would result in rendering this current less easily detectable by shock or magnetisation, but make no difference in its effect on a galvanometer. Aided by this thought, he repeated Faraday's experiment with the balanced coils referred to in § 8 ("Experimental Researches," Vol. I., § 1,790 *et seq.*). A galvanometer was provided having two equal wires of the same length and thickness wound on the same frame, and also a double magnetising spiral was prepared by winding two equal wires round the same piece of hollow straw. Coil No. 1, connected with a battery, was supported perpendicularly on the table, and coils Nos. 3 and 4 were placed parallel, one on each side, and each coil connected in series with one coil of the differential galvanometer and with one spiral of the magnetising helix. The two outside coils were then adjusted so that when the battery circuit was made and broken, and the current started and stopped in the middle coil, no indication was given by the galvanometer, and no magnetisation produced in a steel needle placed in the double helix. A thick zinc plate was then introduced between the primary coil and one of the secondaries, and it was found that the needle of the galvanometer still remained stationary on making and breaking the primary current, but that the steel needle in the spiral became powerfully magnetic. This indicated that the two secondary currents, whilst still equal in total quantity, had been so affected that one had a less maximum value than the other, and hence a differential magnetising action was produced. A similar effect was observed when a galvanometer and magnetising spiral were together introduced into the secondary circuit of a single primary and secondary circuit. The interposition of a metal sheet considerably reduced the magnetising power or the shock, but left the galvanometer deflection unaltered. In order to increase the number of facts, this last experiment was varied by the exchange of a soft iron needle for the hard steel needle in the magnetising coil, the metal screen being interposed in each case, and it was found that whereas the metal screen cut off almost entirely the power of the secondary current to magnetise hard steel, it could yet slightly magnetise the soft iron. A screen of cast iron, half an inch thick, however, not only neutralised the power to magnetise hard steel, but reduced the deflection of the galvanometer as well. The general explanation of the foregoing facts, as due to Henry, is as follows:—The secondary current, as we have seen, is a brief discharge, which rises very suddenly to its

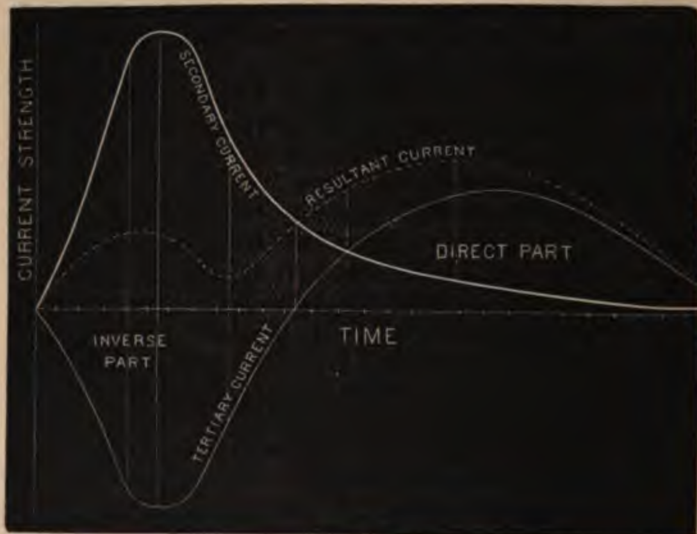

Fig. 25.

maximum value and then fades gradually away. The current curve of the secondary current, due to the rupture of a primary circuit, may be represented by the thick firm line in Fig. 25. If a metal screen is interposed between the primary and the secondary circuit the screen gets a similar secondary current generated in it, and this last again acts by induction to generate a tertiary current in the secondary circuit. This tertiary current consists of two portions—first, an inverse part opposite in direction to the secondary current in the screen, and, secondly, a succeeding direct current. Let the current curve of this tertiary current in the secondary circuit be represented by the fine firm line in Fig. 25. The total quantities of electricity flowing in each part of the two portions of the tertiary current are equal. The resultant effect, then, of the action of the primary current when interrupted is to cause in the secondary circuit the true secondary current, which is an unidirectional flux (thick curve), and a superimposed tertiary current, which is a bi directional flux, its algebraic total of quantity being zero. If we add together at each instant the ordinates of the two current curves we get a resultant curve (dotted line) which represents the actual current curve in the secondary circuit. The total area (electric quantity) enclosed between the horizontal line and the dotted curve must be equal to the total area enclosed between the thick firm line and the horizontal, because we have added and subtracted equal areas; but the *maximum* ordinate of the dotted curve will be less than that of the thick firm line curve, and the *form* of the curve will be very different also. It is, then, clear that the superposition of a complete tertiary current, which is of itself but very little able to affect a galvanometer on a secondary current which gives a definite galvanometer indication, is not able to alter that galvanometer deflection, depending as it does on the total quantity of the discharge. The magnetising power and shock, however, depend upon the maximum value or suddenness with which the induced current rises to its maximum value, and this factor is very much affected by the overlaying of a secondary current by a tertiary. We see, then, that the experiences of Faraday and Henry may be completely reconciled, and that magnetic screening is a phenomenon which in this case depends upon the nature of the detecting instrument in the secondary circuit.

The practical outcome of much of the foregoing discussion of magnetic screening is that the use of lead-covered cable for the conveyance of periodic currents of the usual frequency (60—100 alternations per second) is of *no advantage* in respect of prevention of inductive disturbance in neighbouring telephone wires. Not only is the lead too poor a conductor, but the frequency of alternation is too small to render the magnetic screening effective. The only really satisfactory method of annulling the inductive disturbance is to carry the periodic current along a conductor which lies in the axis of, and is insulated from, a concentric enclosing tube or sheath, which acts as a return. This return must be itself insulated from the earth, and the condition to be fulfilled is that at any instant, and at any section, the algebraic sum of the currents in the core and sheath must be zero; reckoning current in one direction positive, and in the other negative.

The whole question of magnetic screening has been worked out mathematically by several mathematicians, and besides the section in Clerk-Maxwell's Treatise (Vol. II., § 654, 2nd Ed.), the advanced student may be referred to memoirs by Prof. Charles Niven "On the Induction of Electric Currents in Infinite Plates and Spherical Shells" (*Phil. Trans.* Roy. Soc., 1881, p. 307), and also to Prof. H. Lamb "On Electrical Motions in a Spherical Conductor" (*Phil. Trans.* Roy. Soc., 1883, p. 519).

§ 9. Reaction of a Closed Secondary Circuit on the Primary.—If a Bell telephone is placed in series with a coil of many turns of fine wire wound on a hollow bobbin, and if both are placed in series with the secondary circuit of a small induction coil, the strength of the secondary current can be so adjusted that the telephone emits a low murmur or rattle. This being the case, let a solid bar of copper be introduced into the bobbin of fine wire, and it will be found that the noise of the telephone is *increased*. If a bundle of fine iron wires is substituted for the copper rod, it will, on the other hand, *reduce* the noise or stop it altogether. The explanation of this effect is to be found in the reaction which a closed secondary circuit has upon its primary in reducing the impedance of the primary. We have shown in Chapter III. (p. 152) that the re-active effect of the secondary is to increase the resistance and reduce the inductance of the primary circuit, and we have deduced two formulæ given by Maxwell for the value of the equivalent resistance R′ and the equivalent inductance L′ of a primary coil of resistance R and inductance L in the presence of a secondary coil of resistance S and inductance N, the magnetic circuit having a constant resistance, and the mutual inductance being M. Hence the equivalent impedance Im′ of the primary coil in presence of the secondary is $\sqrt{R'^2+p^2L'^2}$, and its isolated or intrinsic impedance is Im equal to $\sqrt{R^2+p^2L^2}$. The question is, which is the greater—Im′ or Im? To discover this, take for R′ and L′ the values given on page 153, and we have

$$R'=R+\frac{p^2M^2S}{S^2+p^2N^2},$$

and

$$L'=L-\frac{p^2M^2N}{S^2+p^2N^2}.$$

Forming from these the function $R'^2+p^2L'^2$, we have

$$R'^2+p^2L'^2=R^2+p^2L^2+\left[\frac{2p^2M^2RS-p^4M^2(2LN-M^2)}{S^2+p^2N^2}\right]$$

or

$$(\mathrm{Im}')^2-(\mathrm{Im})^2=\frac{2p^2M^2RS}{S^2+p^2N^2}-\frac{p^4M^2(2LN-M^2)}{S^2+p^2N^2}.$$

If $S=\infty$, or the secondary circuit is open, the right-hand side of the above equation is zero, and we find that the impedance of the primary circuit is not altered by the presence of the open secondary, as of course it should not be. If S is made very small, the value of the difference of $(\mathrm{Im}')^2$ and $(\mathrm{Im})^2$ continually approximates to the value $-\frac{p^4M^2(2LN-M^2)}{p^2N^2}$.

Now, $LN-M^2$ is always a positive quantity if not zero, and the other quantities in this last fraction are squares. Hence the value of this last fraction is always a positive quantity if not zero. Hence it follows that if Im′ is not equal to Im it is always *less* than it; in other words, the presence of the closed secondary circuit always diminishes the effective impedance of the primary. We have seen (page 153) that the maximum value of the primary current for the induction coil without iron and with simple periodic impressed electromotive force applied to the primary circuit is

$$I=\frac{E}{\mathrm{Im}'},$$

or

$$\text{Maximum current}=\frac{\text{Maximum E.M.F.}}{\text{Effective impedance of primary}}.$$

Hence, since the closing of a secondary circuit around a primary reduces its impedance it must also, if the primary coil is subjected to a constant impressed electromotive force, increase the flow of current through the primary. The explanation of our experiment with the copper rod is now simple. The introduction of the copper rod into the fine wire helix is equivalent to approximating to a primary coil a closed secondary circuit. The impedance of the fine wire circuit to the alternating current from the secondary circuit of the induction coil is hence reduced; it gets more current, and the telephone is made to emit a louder sound. If, however, a core of divided fine iron wire is introduced into the fine wire helix, the result is simply to increase the impedance of that circuit, and therefore to reduce the current actuating the telephone. When considering in particular the theory of the induction transformer as applied to electric distribution we shall see the above principles have important practical bearings.

In a paper recording some experimental results on the self-induction and resistance of compound conductors* Lord Rayleigh has given some comparisons of the results of theory and experiment on Maxwell's formulæ above alluded to. By the use of a resistance and inductance bridge, very similar to one designed by Prof. Hughes, the measurements of the inductance and resistance of a circuit can be made separately with ease. A pair of wires was wound on one bobbin; each wire had a resistance of nearly ·1 ohm, and a diameter of ·037in. Each coil consisted of nine double convolutions. In certain arbitrary units the resistance of one of these copper wires to steady currents was 1·75, and its inductance 11°·2. The values were obtained when the other coil was on open circuit. On closing the unused coil, the resistance of the first rose to 2·67 and its inductance fell to 4°·7.

To compare this with the theory.

The formulæ are
$$R' = R + \frac{p^2 M^2 S}{S^2 + p^2 N^2}$$
$$L' = L - \frac{p^2 M^2 N}{S^2 + p^2 N^2}.$$

Now $R = S = 1{\cdot}75 \times {\cdot}0492 \times 10^9$ absolute C.-G.-S. units of resistance,

and $L = N = 11^\circ{\cdot}2 \times 1553$ centimetres,

$M = 11^\circ \times 1553$ centimetres,

and $p = 2\pi n = 2 \times 3{\cdot}1415 \times 1050.$

* See *Phil. Mag.*, December, 1886, p. 469.

The periodic current used had a frequency of 1050 per second;

hence, $$\frac{p^2 M^2}{R^2 + p^2 L^2} = \cdot 6.$$

Therefore $$R' = R\,(1 + \cdot 6) = 1\cdot 6\ R,$$

and $$L' = L\,(1 - \cdot 6) = \cdot 4\ L;$$

but $$1\cdot 6 \times 1\cdot 75 = 2\cdot 8 = R',$$

and $$\cdot 4 \times 11^{\text{s}}\cdot 2 = 4^{\text{s}}\cdot 5 = L'.$$

These calculated values compare very favourably with the observed values, viz.:

$$R' = 2\cdot 67,\ L' = 4^{\text{s}}\cdot 7;$$

and experimentally confirm the truth of Maxwell's formulæ for the increased resistance and diminished inductance of a circuit when placed near a closed secondary circuit.

§ 10. **Hughes's Induction Balance and Sonometer.**—In 1879 Prof. Hughes constructed and described a very perfect induction balance, with which he was able to conduct researches of an exceedingly interesting character. In order to have a perfect induction balance, he found it necessary to make all the four coils exactly similar.* Four boxwood bobbins (*see* Fig. 26) are each wound over with 100 metres of No. 32 copper wire. These coils are arranged in pairs at a considerable distance apart, so that the coefficient of mutual induction between the separated pairs is negligible. Two of the coils, A and B, are joined in series with each other and with a battery and interrupter I, and the other two coils, C and D, are employed respectively as secondary coils to these two. These secondary coils are in series with each other and with a telephone receiver T, and are so joined up that the direction of the induction of A on C is opposite to that of B on D. One pair of coils is placed in a fixed position, and the other pair can be slightly moved to or from each other by means of a micrometer screw. The coils are first

* "On an Induction Current Balance." By Prof. D. E. Hughes. *Proc. Roy. Soc.*, No. 196, May 5th, 1879.

adjusted so that the inductions are equal and opposite, and on listening at the telephone the opposing secondary currents produce at best but a very slight sound, which can be perfectly abolished by adjusting the distance of one pair of coils. When this is the case, if we insert in the opening of the bobbin of one of the primary coils a disc or piece of metal *d*, the balance is destroyed, and we hear sounds more or less intense. In order to get some comparative measurements, Prof. Hughes designed a companion instrument, called a *sonometer*. In this instrument a pair of primary coils are, as before (*see* Fig. 27), joined in series with each other and with a battery. The coils are fixed

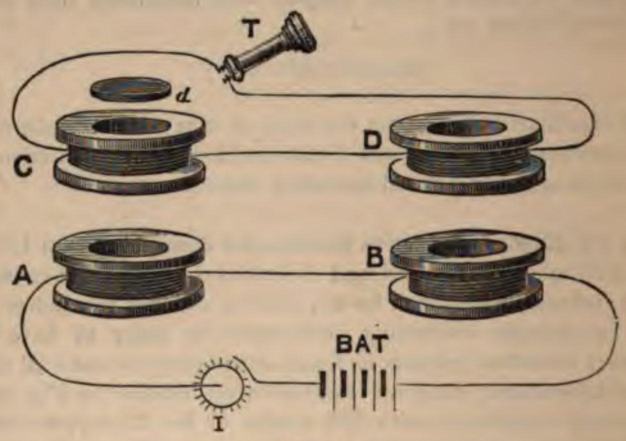

Fig. 26.

at the extremities of a bar. Between these primary coils slides a single secondary coil, and the primary coils are so wound that their inductions on this secondary coil are equal and opposite. When this secondary coil is exactly between the two primary coils, a telephone placed in series with the secondary coil gives out no sound when the primary current is rapidly interrupted. If, however, the secondary coil is slid from one primary and towards the other, the differential action creates an induced current detected by the telephone. By reading off on the bar the extent of displacement necessary to create in the telephone a sound of a certain magnitude an arbitrary reading can be obtained corresponding to every different value of the secondary current. A switch is provided by means of which the same telephone can be shifted rapidly from the induction balance secondary circuit to the sonometer secondary circuit. The experiments first performed consisted in placing within one primary coil of the induction balance certain equal-sized discs of different metals, and then so arranging the sonometer secondary coil that the noise in the telephone produced by the current in the secondary of the sonometer was judged by the ear to be equal to the sound produced in the telephone when it was shifted to the secondary circuit of the induction balance, and in which the inductive balance had been broken down by the insertion of the disc of metal. Discs of various metals the size and shape of an English shilling were made, and when inserted in the induction coil, the

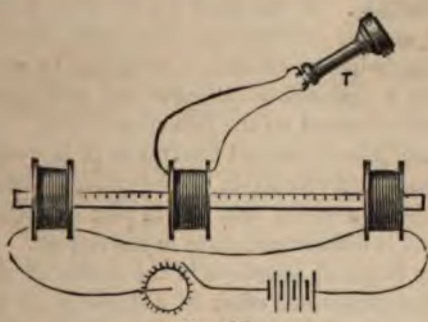

FIG. 27.

sonometer bar readings, reckoned from the centre or absolute zero of sound given in certain arbitrary degrees, were as follows :—

Silver (chemically pure)	125	German Silver	50
Gold	117	Iron (pure)	40
Silver coin	115	Copper (alloy)	40
Aluminium	112	Lead	38
Copper	100	Antimony	35
Zinc	80	Mercury	30
Bronze	76	Bismuth	10
Tin	74	Zinc (alloy)	6
Iron (ordinary)	52	Carbon	2

This list does not agree in order entirely with that of any of the lists of electrical conductivity. In some degree it evidently has reference to conductivity, because, roughly speaking, the best conductors come at the top and the worst at the bottom ;

but whilst it is headed by silver, which has the highest conductivity per unit of volume, we find aluminium, which has the highest conductivity per unit of mass, occupying a position above that of copper. The disturbing effects of the metal on the induction balance is not, however, simply proportional either to the conductivity per unit of mass or per unit of volume. In more recent experiments a graduated zinc wedge pushed in more or less between one pair of coils of the induction balance was employed to obtain comparative numbers representing the disturbance produced when discs of various metals are inserted in the other coil. The elementary theory of the induction balance is of course contained in all that has gone before in this and the last chapter. It is, generally speaking, dependent for its action on effects similar to those producing magnetic shielding. If the discs are slit so as to prevent circumferential electric currents in their mass, their action in disturbing the inductive balance is mitigated or annulled. If the metal disc is replaced by a copper coil with open extremities no effect is observed on the inductive balance. If the ends of the coil are joined, the coil behaves as if it were a metallic disc and causes loud sounds in the telephone. The effect due to the iron disc is a mixed one. It in part acts like any other metal disc, but it differs from them in one respect. If any non-magnetic disc is placed edgeways in the centre of the primary bobbin it has a diminished effect in disturbing the balance; in the case of iron the disturbance is increased by turning the disc edgeways. In order to have before us a typically simple case, imagine an induction balance made of two very long primary helices and each embraced near the centre by a small secondary coil. Let the primary coils be traversed by a simple periodic current. We have then in the interior of the primary coil a uniform magnetic field varying synchronously with the primary current in a simple periodic manner, and the rate of change of the magnetic field at any instant will be a measure of the electromotive force acting in the secondary circuit. Suppose into one primary helix is inserted a thin copper tube; this will form a closed secondary circuit, and secondary periodic currents will be induced in it, flowing round the cylinder in directions parallel to the turns of the primary helix. As this copper cylinder possesses a very sensible *time constant*, the phase of these secondary currents in the copper cylinder will be nearly opposite to that of the primary current. The resultant magnetic field in the interior of the cylinder is therefore that due to the resultant of these two simple periodic currents which are nearly opposed in phase. Hence the absolute magnitude of the interior field and its rate of variation will be less than if the copper cylinder was removed. It results, therefore, that the induction through the secondary helix and the electromotive force impressed on it will be diminished by the presence in the primary coil of this copper cylinder. The diagram given on page 148, showing a geometrical construction for the magnitudes of the primary and secondary currents in an induction coil without iron, shows us why the primary and secondary currents are thus more or less opposite in phase. Since, in a general way, the higher the conductivity of the tube or disc introduced into the primary the greater the time constant and the greater the lag in phase of the currents induced in this metallic circuit behind the phase of the inducing primary, it follows that the resultant interior field, acting to produce inductive electromotive force in the secondary helix, will be diminished by the introduction of discs of very high conductivity more than by discs of very low conductivity.

From the principles discussed under the head of magnetic shielding, it would appear that the differences between various metals inserted as discs in the induction balance would be *less* marked at very high speeds of interruption than at very low ones. With respect to the action of iron, two effects have to be considered which are the result of very different actions. The introduction of the iron into the primary coil reduces the magnetic resistance of the circuit of induction of that coil, and this cause, if it operated alone, would destroy the inductive balance by *raising* the inductive electromotive force in that secondary circuit corresponding to the primary into which the iron is introduced; but the iron disc, like every other disc, gets circumferential induced currents created in it, and these, if they acted alone, would destroy the inductive balance by *lowering* the inductive electromotive force in that secondary coil.

These two effects conflict, and it is an interesting confirmation of theory to find that Prof. Hughes says it is possible to introduce into one primary coil of the induction balance a disc of iron and some soft iron wires in such positions that these opposite actions nullify each other, and though each mass of iron separately would destroy the induction balance, yet the two together being introduced, complete silence in the telephone is the result. The sensibility of the induction balance to minute differences of electric conductivity and magnetic permeability is very remarkable. If into one coil of a carefully-adjusted balance we place a good sovereign, or shilling, and into the other a bad one, the telephone detects the base coin with unerring certainty by the loud noise given out. In the same way, if two pieces of soft iron are introduced into the two primary coils, and a balance is obtained, the mere magnetisation of one of them will be at once detected, because that

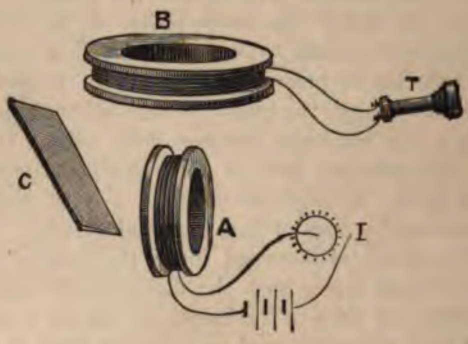

FIG. 23.

magnetised piece becomes thereby less permeable, and destroys the balance. We may present the rough general theory of the induction balance in another way. Let the "coin" be simply regarded as a closed circuit, between which and the primary circuit surrounding it there is a certain coefficient of mutual induction. The two primary coils forming one primary circuit have, on the whole, no action on the two secondary coils, forming one secondary circuit, and we may therefore consider the primary circuit as if it were in a position *conjugate* to the secondary. The coin, however, is acted upon inductively by the primary circuit, and the eddy currents or secondary currents generated in it react on the secondary circuit, causing in it tertiary currents, which affect the telephone. Looking at it from this point of view, we might construct an induction balance thus. Let A (*see* Fig. 28) be a single primary coil, and B a secondary coil, having a telephone in series with it. Place the coil B in a position conjugate to A—that is, with its axis at right angles to that of A. Then let variation of current in A produce no current in B. Now hold a sheet of copper anywhere, say at C, and the telephone will be caused to sound. For A, though it cannot affect B inductively directly, yet it can produce a secondary current in C held at a non-conjugate position, and these secondary currents in C will create other tertiary currents in B. The experiment will appear to indicate a sort of *reflection* of inductive power.*

This was experimentally shown by Mr. Willoughby Smith in his Paper on "Volta-Electric Induction" (see *Journal of Society of Telegraph-Engineers*, Vol. XII., p. 465).

An interesting experiment of Mr. Willoughby Smith's in induction is to employ a simple Bell telephone receiver, unconnected with any circuit, as an induction finder. If a coil is traversed by a primary current, rapidly intermittent or alternate, then a Bell telephone held anywhere in the new field emits a sound. The pulsating field disturbs the magnetism of the telephone magnet, and enables this, therefore, to detect rapid electromagnetic disturbances at the place where it is. The induction balance, combined with a telephone, is an apparatus of extreme sensitiveness. It renders evident the smallest differences of weight, nature, degree of purity or temperature of two conductors of identical dimensions, such as two coins placed in identical conditions in respect of the two systems of coils.

It enables us to detect very small masses of metal in a badly conducting body, and may be employed with much advantage in verifying the insulation of the different windings of a coil, the ends of which are open. At the same time it lends itself better

* The full theory of the induction balance has been given by Prof. Oliver Lodge. See *Proc. Phys. Soc. London*, Vol. III., p. 187. Also in the same volume is a note by Prof. J. H. Poynting "On the Graduation of the Sonometer."

to qualitative than to quantitative work, as it is difficult to interpret rigorously the results obtained.*

§ **11. Transmission of Rapidly Intermittent, Alternate, or very brief Currents through Conductors.**—Some experiments of Prof. Hughes on the self-induction of metallic wires have been the means of directing attention of late more closely to the nature of the propagation of currents of rapid periodicity through metallic conductors, and although mathematical writers, particularly Mr. Oliver Heaviside, had considered the problem, yet these experimental results drew the attention of many to

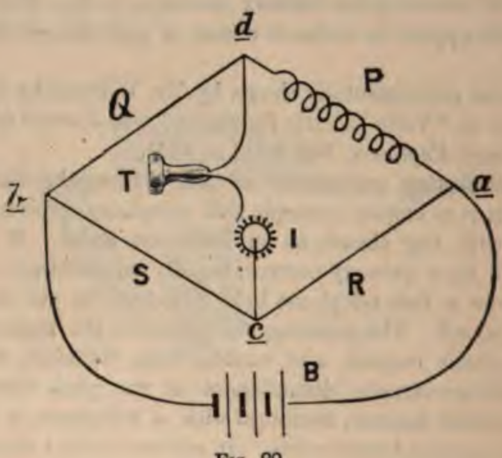

FIG. 29.

this question to whom the more recondite investigations were unknown. Prof. Hughes's experiments† on the self-induction

* For further information on the use and theory of the induction balance the student may consult Mascart and Joubert's *Electricity*, Vol. II., §986; also Hughes' *Phil. Mag.* [5], Vol. II., p. 50, 1879. On the differential telephone, *see* Chrystal, *Phil. Trans. Roy. Soc. Edin.*, Vol. XXIX., p. 609, 1880. O. Lodge, *Proc. Physical Soc. London*, Vol. III., p. 187, on intermittent currents and the theory of the induction balance.

† These experiments formed the subject of Prof. Hughes's Inaugural Discourse to the Society of Telegraph-Engineers on the occasion of his election to the office of President. See *Journal Society Telegraph Engineers*, January 28, 1886, "The Self-Induction of an Electric Current in Relation to the Nature and Form of its Conductor."

of metallic wires were made with a combined resistance and induction bridge of somewhat novel form. Suppose that a quadrilateral be formed of four conductors P, Q, R, S, only one of which, P, has any sensible self-induction, and let the diagonals be completed by a telephone T, and battery B, with interrupter I. In the first place, let the resistance-balance be obtained for steady currents. This can be achieved by placing the telephone with the interrupter as a conjugate circuit to the battery (*see* Fig. 29), and shifting one resistance, R, until a balance is

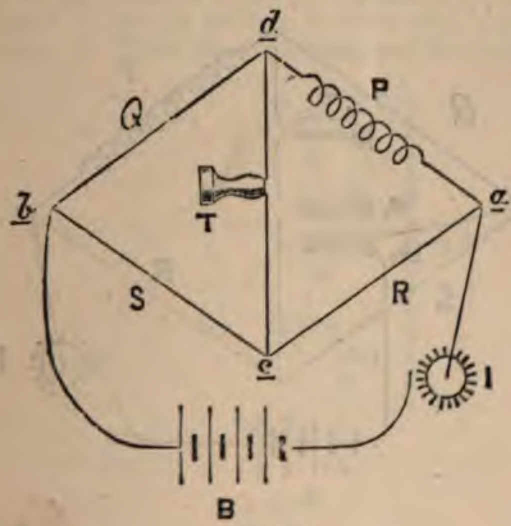

FIG. 30.

obtained. By a suitable adjustment complete silence can be obtained in the telephone.

Next let the interrupter be removed to the battery circuit, and all other arrangements remaining the same (*see* Fig. 30) it will be found that the balance is destroyed, and that no mere change in the value of the resistance R will enable a perfect balance to be obtained. The reason for this is that on closing the battery circuit the inductance of P introduces a counter electromotive force into P and the potential rises at *c* faster than at *d*, and on breaking the circuit the potential at

c dies down faster than at d; and hence at each make and break the telephone is subjected to an alternate flux of current, and this causes it to emit a sound. Supposing that an attempt is made to get rid of this sound by shifting the point c so as to alter R, the steady balance will be destroyed and the telephone will be traversed by a current during the time when all the currents have become steady; but no such change in the value of R will prevent a *variation* of current taking place through the telephone during the complete period from the first instant

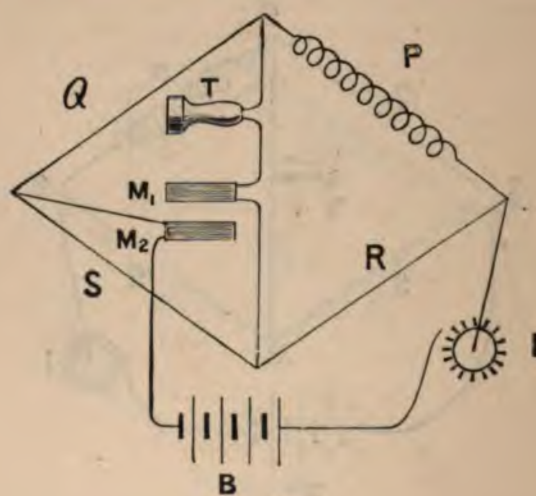

FIG. 31.

when the battery circuit is closed to the instant when it is opened again.

The only way in which a balance can be obtained in this last arrangement is by introducing into the telephone circuit an electromotive force which shall be capable of being made at every instant to balance the inductive electromotive force due to the inductance of P. Prof. Hughes does this very ingeniously by introducing a pair of mutually inductive coils into the battery and telephone circuits, and the final arrangement is in Fig. 31. M_1 and M_2 are a pair of coils, one of which, M_2, is in the battery circuit and is fixed, and the other, M_1, is in the telephone circuit, and can be placed so that whilst its centre coincides with that of M_2 its axis makes any required angle with that of M_1. In this way the mutual inductance between M_1 and M_2 can be varied from zero when the coil axes are at right angles to a definite maximum value when they are co-linear.

It is, however, found that when the coils M_1 M_2 are in certain positions that the inductive electromotive force set up in the telephone circuit by the induction of M_1 on M_2 can be made to neutralise the impulsive electromotive force due to the inductance of P, when, in addition, a certain value is given to the resistance R. Under these circumstances the bridge can

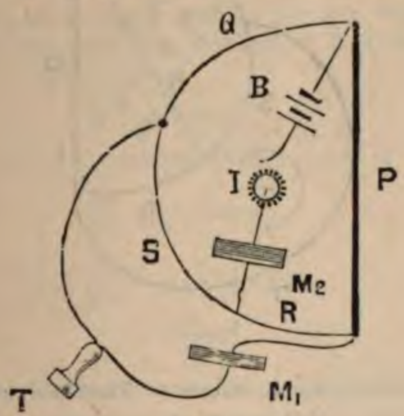

FIG. 32.

be balanced and the telephone completely silenced, both when the interrupter is in the battery circuit and in the telephone circuit; in other words, can be balanced both for steady and for variable currents.

In the arrangement adopted by Prof. Hughes the resistances Q, R, and S, were sections of one and the same fine German silver, 1 metre long, and having a total resistance of 4 ohms (*see* Fig 32). The ends of this wire were joined to the conductor under investigation P, and the rest of the apparatus was arranged as described.

In order to investigate the relation between the resistances and inductances which holds good when the bridge is balanced for steady and for variable currents, let us draw a diagram (Fig. 33) representing the network of conductors, and call the current at any instant in the inductive branch P, x, and that in the branch Q, y, and that in the telephone circuit z. The current in the battery branch is then $x+y$. Let L be the inductance of P, and M the mutual inductance of the coils placed in the circuit B and T, and let all other circuits, Q, R, and S, have no sensible inductance. Let e be the electromotive

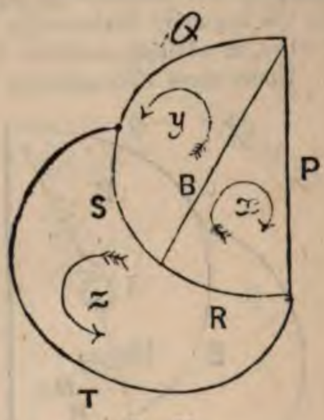

FIG. 33.

force of the battery at any instant t. Then the currents in the various branches at that instant are as follows:—

In the	branch	P	current	is	x
"	"	R	"	"	$x+z$
"	"	S	"	"	$y-z$
"	"	Q	"	"	y
"	"	B	"	"	$x+y$
"	"	T	"	"	z

Applying Kirchhoff's corollaries to each of the three meshes of the network we have three equations, viz.,

$$\text{(i)}\quad \mathrm{P}x+\mathrm{B}\overline{x+y}+\mathrm{R}\overline{x+z}=e-\mathrm{L}\frac{dx}{dt}-\mathrm{M}\frac{dz}{dt}$$

$$\text{(ii)}\quad \mathrm{B}\overline{x+y}+\mathrm{S}\overline{y-z}+\mathrm{Q}\,y=e-\mathrm{M}\frac{dz}{dt}$$

$$\text{(iii)}\ \mathrm{R}\,\overline{x+z} + \mathrm{T}\,z - \mathrm{S}\,\overline{y-z} = -\mathrm{M}\frac{d\,\overline{x+y}}{d\,t},$$

and these three equations contain the means for finding at any time t the current in any branch.* If we suppose the bridge to be balanced for variable currents then z is zero, and on making this limitation we find the above equations reduce to the two,

$$\mathrm{Q}y - \mathrm{P}x - \mathrm{L}\frac{dx}{dt} = \mathrm{R}x - \mathrm{S}y \quad . \quad . \quad . \quad (\mathrm{A})$$

and

$$-\mathrm{M}\frac{dx}{dt} - \mathrm{M}\frac{dy}{dt} = \mathrm{R}x - \mathrm{S}y \quad . \quad . \quad . \quad (\mathrm{B})$$

Furthermore, let us assume that the currents vary according to a simple periodic law. In this case, if X is the maximum value of x, then we can write

$$x = \mathrm{X}\sin p\,t,$$

where p as usual $= 2\,\pi\,n$, n being the frequency of the alternation, hence

$$\frac{d\,x}{d\,t} = p\,\mathrm{X}\cos p\,t$$

$$\text{and}\ \frac{d^2x}{d\,t^2} = -p^2\,\mathrm{X}\sin p\,t$$

$$= -p^2\,x.$$

Adopting the fluxional notation, it is convenient to write $\dot{x}$ for $\frac{dx}{dt}$ and $\ddot{x}$ for $\frac{d^2x}{dt^2}$. Hence, for simple periodic variation of a current x, we always have the condition

$$-\ddot{x} = p^2\,x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{C})$$

If we differentiate the two equations (A) and (B), and eliminate $\ddot{x}$ by the help of (C), we get the following two equations (E) and (F), which, together with the original two (A)

* The general method of finding the current equations for any network is given in Maxwell's "Treatise on Electricity," 2nd Edition, Vol. II., § 755. *Also see* "Problems on Networks of Conductors," by J. A. Fleming, *Phil. Mag.*, September, 1885, Vol. XX., p. 221; or *Proceedings Phys. Soc., Lond.*, 1885.

and (B), give us the necessary four for eliminating the variables. We have

$$Q\,y - P\,x - L\,\dot{x} \;= R\,x - S\,y \;.\;\;.\;\;.\;\;.\;\;\text{(A)}$$

$$-M\,\dot{x} - M\,\dot{y} \;= R\,x - S\,y \;.\;\;.\;\;.\;\;.\;\;\text{(B)}$$

$$Q\,\dot{y} - P\,\dot{x} + L\,p^2\,x = R\,\dot{x} - S\,\dot{y} \;.\;\;.\;\;.\;\;.\;\;\text{(E)}$$

$$M\,p^2\,x + M\,p^2\,y \;= R\,\dot{x} - S\,\dot{y} \;.\;\;.\;\;.\;\;.\;\;\text{(F)}$$

The student who has mastered the elements of determinant analysis will recognise that the variables x, y, $\dot{x}$, $\dot{y}$ can be eliminated from these equations, and the relation which must always hold good between the constants can be found by equating to zero the determinant of these four equations; we have, then

$$\begin{vmatrix} -L, & 0, & -(P+R), & (Q+S) \\ -M, & -M, & -R, & S \\ -(P+R), & (Q+S), & L\,p^2, & 0 \\ -R, & S, & M\,p^2, & M\,p^2 \end{vmatrix} = 0$$

This determinant writes out into the sum of three terms, viz. :—

$$p^2\,L\,[L\,(M^2\,p^2 + S^2) - M\,(S^2\,S\,Q + R\,Q + S\,R)] +$$
$$+ (P+R)\,[M^2\,p^2\,(P+Q+R+S) - S\,(Q\,R - S\,P)] +$$
$$+ (Q+S)\,[M\,L\,p^2\,(R+S) + R\,(S\,P - R\,Q) - M^2\,p^2\,(P+Q$$
$$+ R + S)] = 0$$

In order that the sum may be zero, each factor in the square brackets must be separately zero, and it will be found on reduction that these factors equated to zero are equivalent to the two equations :—

$$Q\,R - S\,P = M\,L\,p^2 \;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;\text{(G)}$$

and $$M\,(P+Q+R+S) = S\,L \;.\;\;.\;\;.\;\;.\;\;\text{(H)*}$$

These equations express the relation which holds good between the resistances of the branches and the self and mutual induction coefficients of a Hughes bridge when the bridge is balanced for variable currents.

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* These equations were given by Lord Rayleigh in the discussion on Prof. Hughes's Paper. See also Lord Rayleigh "On the Self-Induction and Resistance of Compound Conductors," *Phil. Mag.*, Dec. 1886, p. 471. Equivalent equations have been also arrived at by Prof. H. F. Weber and Mr. Oliver Heaviside.

It will be seen that the ordinary relation of the resistances for steady balance, viz., P : Q = R : S is departed from, and that we have for the resistance of branch P to variable currents the value

$$P = \frac{QR - MLp^2}{S} = \frac{QR}{S} - \frac{MLp^2}{S},$$

and for the inductance of branch P the value

$$L = \frac{M(P+Q+R+S)}{S}.$$

In some of his experiments Prof. Hughes interpreted his results on the assumption that P was always equal to $\frac{QR}{S}$, and L was equal to M, but the complete investigation shows that this is not the case. A very full theoretical and practical examination of the induction bridge has been given by Prof. H. F. Weber, for which the student is referred to the pages of the *Electrical Review*, Vol. XVIII., p. 321, 1886, and Vol. XIX, p. 30, 1886.*

The whole method of construction and usage of the induction bridge has been the subject of elaborate examination by Lord Rayleigh in a Paper on the self-induction and resistance of compound conductors (*Phil. Mag.*, December, 1886), from which we shall quote freely in what follows. Discarding the tooth-wheel interrupter, as it does not give a regular valuation of current corresponding in period to the passage of a tooth, Lord Rayleigh used a harmonium reed, the vibrating tongue of which made contact once during each period with the slightly-rounded end of a brass or iron wire advanced exactly to the required position by means of a screw cut upon it. Blown with a regulated wind, such reeds are capable of giving interruptions of current of about 2,000 per second. The one usually employed had a frequency of 1,050 vibrations per second. The induction compensator consisted of two circular coils, one of which was fixed and the other movable round an axis, so placed that the flat circular coils could be placed either with their planes coincident or at right angles. If the inner coil is very small compared with the other, and the coils are placed with

* *See also* Mr. Oliver Heaviside in the *Phil. Mag.*, August, 1886.

centres coincident and axes inclined at any angle, θ, and if M_0 be the maximum mutual inductance, and M the inductance in any position, θ, then

$$M = M_0 \cos \theta.$$

This law is, however, not followed when the coils are sensibly of the same size. In this case Lord Rayleigh has shown that the mutual induction is very approximately proportional to the angle between the axes of the coils for a range between 40° and 140°. In the actual experiments the mutual inductance of the coils was determined for each degree of angular displacement of the axes by comparing it with the calculable coefficient between two wires, wound in measured grooves, cut in a cylinder, and it was found that every degree of movement of the movable coil, when the axes were not far removed from perpendicularity, was equal to 776·3 centimetres of mutual induction, the maximum when $\theta = 0$ being 56,100 centimetres. The first example given in the Paper referred to is one on the self-induction and resistance of a coil of copper wire. In the bridge used the resistances Q + R + S were together 4·00 ohms. Resistances were, however, measured in scale divisions of the bridge wire, each one equal to $2{\cdot}04 \times 10^6$ centimetres per second. The copper coil being balanced on the bridge, it was found that the readings of the three resistances and of M were as follows :—

$$Q = 610, \; R = 190, \; S = 1{,}160,$$
$$M = 36^\circ = 36 \times 776 \text{ centimetres},$$

and the frequency n of the vibrations = 1,050, hence $p = 2\pi \times 1{,}050$. Taking the equation (G) and (H) on page 236 and eliminating L, we have for the value of P the equation

$$P = \frac{Q.R}{S}\left[\frac{1 - \dfrac{p^2 M^2 (Q + R + S)}{S.\,Q.\,R}}{1 + \dfrac{p^2 M^2}{S^2}}\right].$$

Substituting the values above, we find

$$P = {\cdot}876 \frac{QR}{S} = 87{\cdot}5 \text{ scale divisions.}$$

This gives the value of the real resistance of P to the periodic currents, and we see that if we neglected the peculiarity of the bridge and simply assumed the ordinary law, that the resistance of P was equal to Q R ÷ S, we should make an error of some 12 per cent. On actually balancing the bridge for steady currents the resistance of P was found to be 87·3 scale divisions, thus indicating that for this copper coil at the frequency used the resistance to variable currents was the same as to steady ones.

On inserting a solid copper rod into the coil and measuring again the resistance and self-induction it was found that the values of the reading were Q = 660, R = 190, M = 29°·5, instead of as before, Q = 610, M = 36°. Hence the introduction of another closed secondary circuit (viz., the copper rod) increased the real resistance and diminished the real self-induction in accordance with the principles explained on page 153, at which place we demonstrated Maxwell's equation for the increased resistance and diminished self-induction of a primary circuit when in contiguity to a closed secondary circuit. The next example selected was that of a soft iron wire, 160 centimetres long and 3·3 mm. dia. Here, with the variable currents from the reed interrupter of the same period as before, a balance was obtained for

$$Q = 178 \qquad R = 190 \qquad S = 1{,}592$$

M = 8 × 776 centimetres, from which we find

$$P = \cdot 985 \frac{Q.R}{S} = 20 \cdot 93 \text{ scale divisions.}$$

The resistance to steady currents of the same wire was

$$P_0 = \frac{100 \times 190}{1{,}670} = 11 \cdot 38 \text{ scale divisions.}$$

Hence the effective resistance to variable currents of a frequency 1,050 was 1·84 times the resistance to steady currents. Here we have presented to us the phenomena characteristic of the behaviour of conductors to electric currents rapidly intermittent or reversed. The real resistance of the conductor is increased. This is not to be confused with the fact that for intermittent currents the impedance ($\sqrt{R^2 + p^2 L^2}$) measured in ohms is greater than the ohmic resistance (R); but it is to be understood as a real increase in the rate at which energy is dissipated per unit of current, and there are many lines of thought which lead to the conception that such increase of resistance is due to the fact that the current density for such periodic currents is not uniform over the cross-section of the wire, but the current density is greatest along the outer layers of the wire. Hence, under rapidly periodic currents the inner portions of a conducting wire are nearly deserted by, or rather are never reached by, the currents, and, as far as current carrying duty is concerned, might as well be away. One might represent this notion by a very coarse imagery, thus: Let relative density of current or quantity passing per second through unit of cross-section of a conductor per unit of time be represented like relative density of population by degree of density of shading. Then the flow of a steady current

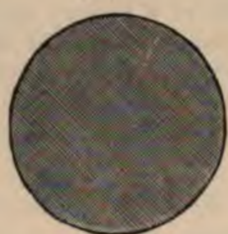

FIG. 34.

FIG. 35.

through the section of a wire might be represented as in Fig. 34; and the flow of current over the cross-section when the current is rapidly periodic might be represented as in Fig. 35.

Certain lines of thought indicate that we can consider that the current in beginning in a conductor starts its flow first on the outside, and soaks or penetrates inwards into the deeper layers by degrees. We see that on this hypothesis if the current is reversed in sign, or rapidly intermitted, it will not have time to soak or diffuse very far into the mass of the conductor before it is, so to speak, re-called, and its operations will be confined to the outer layers. This is a rather broad way of stating some modern views which are beginning to be held on the *modus operandi* of current flow. According to these views the current in a wire is not got up by a process analogous to starting a flow of water in a pipe by a push applied one end, but it is put into the wire at all points of its surface by energy absorbed from the surrounding dielectric. Other things being equal, the rate at which this equalisation of current across the cross-section of the conductor goes on will be a function of the magnetic permeability of the material. The current in flowing along a magnetisable circuit magnetises it circularly. This magnetisation involves work, and the impressed electromotive force which is increasing the current has to do work, not only against that which may be called the formal inductance of the circuit, or against that part of the counter electromotive force of induction which depends on the form of the circuit, but has to create this circular magnetisation. By keeping to the outer layers of the conductor the periodic current avoids magnetising the deeper layers of the material. By this assumption as to the arrangement of the current under rapid intermittance or alternation we are able to offer a consistent theory of the real increase of resistance which we find for such currents. The inner core or central portions of the wire are as good as not used by the current, and as far as conducting it goes might as well be absent; hence the solid conductor does no more, or not much more, in the way of carrying the current than a hollow or tubular conductor would do, and, accordingly, the real or ohmic resistance of the conductor for such variable currents is greater than it is for steady ones.

Another way of regarding this inequality of current distribution over the section of a wire is as follows:—The counter electromotive force arising from self-induction is greater at the axis or central portion of the wire than it is near the surface. If we consider the whole current flowing across any section of the conductor as made up of little streamlets of currents flowing parallel to each other, the central streamlets or filaments of current experience more opposition in reaching full magnitude than do the outer ones, because of the mutual induction by those surrounding them. The current therefore arrives at its maximum value at the surface of the conductor before it does at the deeper or central portions. If the current is periodic or transitory the central streamlets or current filaments are always greatly inferior in strength to those at the surface. There is reason, then, to believe that a sudden rush of current, very brief in duration, such as the discharge from a Leyden jar or condenser, moves chiefly along the surface of a discharging wire, and the same statement holds good for very rapid pulsatory or alternate currents. Although it may be said that the general principle of this behaviour of variable currents was virtually given by Maxwell,* it has been subsequently carefully developed by Mr. Oliver Heaviside and Lord Rayleigh, and has been brought to the notice of practical electricians chiefly by the experiments of Prof. Hughes above alluded to, and this increase of the resistance proper of a wire for rapidly periodic currents is one of the most striking of the results of his researches. The full mathematical development of the problem, even for comparatively simple cases, leads to some very complex mathematical expressions. Lord Rayleigh has, however, treated with great fulness† one or two cases of practical importance. If R and L are the true ohmic resistance and inductance of a cylindrical straight wire of length l and magnetic permeability μ, to steady currents or currents of very slow alternations, and if an alternate current of simple periodic variation and frequency n is sent through it, then the resistance is increased to R^1 and the inductance diminished to L^1 in such wise that if $p = 2\pi n$, as usual, we have

$$R^1 = R\left[1 + \frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2} - \frac{1}{180}\frac{p^4 l^4 \mu^4}{R^4} + \ . \ . \ \&c.\right]$$

and $$L^1 = l\left[A + \mu\left(\frac{1}{2} - \frac{1}{48}\frac{p^2 l^2 \mu^2}{R^2} + \frac{13}{8640}\frac{p^4 l^4 \mu^4}{R^4} \ . \ .\right)\right]$$

A being some constant depending on the position of the return wire.

These formulæ express the fact that the resistance is increased and the inductance diminished in proportion as the

* Maxwell's "Electricity," Vol. II., § 689-690. In this paragraph it is shown that the counter electromotive force of self-induction at any point in a conductor is a function not only of the time but of the position of the point considered, and varies over the cross section of the conductor.

† "On the Self-Induction and Resistance of Straight Conductors," *Phil. Mag.*, May, 1886., p. 382.

frequency of alternation gradually increases from zero to infinity.

At slow rates of alternation the chief opponent with which the impressed electromotive force has, so to speak, to contend is the ohmic resistance; and the distribution of current across the cross-section of the conductor under these conditions is such as to make that resistance a minimum, and this is known to be so when the distribution is a uniform distribution. The current is then taking the greatest advantage of the conductor, and the heat generated and dissipated per unit of time is less under these conditions than if the same total current were distributed in any other way over the cross-section of the conductor. This last statement can be easily proved. Let the cross-section of the conductor, supposed to be a cylindrical wire, be divided into two equal zones by a circular line. Let the resistance per unit of length of the conductor be r for each portion corresponding to the outer and inner zone. Call the outer portion the sheath and the inner the core of the conductor for brevity. If a total quantity of current, x, flows through the conductor, then the rate of dissipation of energy as heat is $\frac{r\,x^2}{4}$ for each portion per unit of length, or $\frac{r\,x^2}{2}$ for the whole conductor, on the assumption that the current is equally divided between the sheath and the core.

If we suppose the total current, x, to be, however, distributed so that a portion, y, travels by the sheath and the remainder z, travels by the core, then the heat generated per unit of length per unit of time is $r\,y^2$ for the sheath and $r\,z^2$ for the core. Hence, for the equi-distribution of current the energy dissipation is $\frac{r\,x^2}{2} = \frac{r\,(y+z)^2}{2}$, and for the unequi-distribution it is $r\,(y^2+z^2)$. Which, then, is greater, $\frac{r\,(y+z)^2}{2}$, or $r\,(y^2+z^2)$?

Consider the following inequalities:—

$$(y-z)^2 \text{ is greater than } \frac{1}{2}(y-z)^2,$$

or $$y^2+z^2-2\,y\,z \text{ is greater than } \frac{1}{2}(y-z)^2;$$

but $\frac{1}{2}(y-z)^2 = \frac{1}{2}(y+z)^2 - 2yz.$

Hence, $y^2 + z^2 - 2yz$ is greater than $\frac{1}{2}(y+z)^2 - 2yz.$

Adding $2yz$ to both sides, we have

$$y^2 + z^2 \text{ is greater than } \frac{1}{2}(y+z)^2.$$

Accordingly it follows that

$$ry^2 + rz^2 \text{ is greater than } \frac{r}{2}(y+z)^2,$$

or $$ry^2 + rz^2 \text{ is greater than } \frac{r}{2}x^2;$$

that is to say, the rate of energy dissipation is greater for the assumed unequal distribution than for the distribution in which the current is equal in density over the cross-section of the conductor. The same kind of proof may be extended to any other arbitrary distribution of current over the cross-section, and the reasoning will lead to the conclusion that the equi-dense distribution is that which causes the *least* rate of dissipation of energy per unit of current.

For slow alternations the current, therefore, adopts that mode of distributing itself over the cross-section of the conductor which makes the rate of energy dissipation a minimum. On the other hand, for rapid alternation, the current meets with its greatest obstacle from the counter electromotive force of self-induction, and it accordingly distributes itself over the cross-section of the conductor, so as to get as much to the outside as possible, and thus avoids, in the case of magnetic conductors, magnetising the inner layers or portions of the conductor. The endeavour is to make the self-induction a minimum irrespective of resistance. This is only an instance of the broad general principle that behaviour of current for very rapid pulsations, or alternations, is determined by the inductances rather than the resistances, whereas for steady or slowly periodic currents the behaviour is governed by resistance rather than by self-induction.

In order to see under what conditions the alternation of resistance and self-induction becomes sensible, we have to examine the value of the term $\frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2}$ in the above-given series for R^1. We will first take the case of an iron wire ·4 centimetre, say, ·16 inch diameter (No. 8 B.W.G.). The specific resistance of iron in C.-G.-S. measure is about 10^4, so that

$$\frac{R}{l} = \frac{10^4}{\pi \times \cdot 04} = \frac{10^6}{4\pi},$$

$p^2 = 4\,\pi^2\,n^2$, n being the frequency.

Let us take $n = 100$, so that there are supposed to be 100 complete alternations per second. The value of μ is more difficult to assign. For small degrees of magnetisation, and solid iron we may, perhaps, take $\mu = 300$;

then $$\frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2} = \frac{1}{12}\frac{4\,\pi^2\,n^2\,\mu^2\,l^2}{R^2} = \frac{5\cdot 2\,\mu^2\,n^2}{10^{10}}.$$

If $\mu = 300$, $n = 100$, $\mu^2 n^2 = 9 \times 10^8$, and $\frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2} = \cdot 47$ = nearly ·5.

Accordingly for this case $R^1 = R\,(1 + \cdot 47)$ nearly, or the resistance is increased to about half as much again.

If $n = 1{,}000$ we should find $R^1 = 48\,R$, or the resistance would be increased nearly fifty times.

Consider next the case of copper. The specific resistance is 1640 C.-G.-S. units. If a be the radius of the wire in centimetres, then we have

$$\frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2} = \frac{\pi^4}{3}\frac{a^4\,n^2}{(1640)^2} = \frac{1\cdot 2\,a^4\,n^2}{10^5}.$$

If, as before, $n = 100$, this fraction becomes equal to $\cdot 12 a^4$. This shows that for a diameter of one centimetre we should have

$$R^1 = R\,(1 + \cdot 12);$$

and hence for diameters of one centimetre and upwards the resistance of round copper rods becomes very sensibly increased for alternate currents of frequency about 100 per second and upwards. The practical conclusions from the above investigation of importance in electrical engineering are these—*first*, copper rods or conductors should be used, and not iron, for transmitting alternate or intermittent electric currents having a moderate frequency, say, of 100 to 1,000 per second; *secondly*, to avoid, as far as possible, the increase of resistance due to the current keeping to the outer portions of the conductor, the conductor should be in the form of a thin strip, or, better, a tube having walls thin in proportion to the radius. It is to be noted that mere stranding of the conductor, or building it up of separate insulated conductors joined in parallel, will not prevent this augmentation of resistance unless the stranding is of such a kind that portions of the cable which at one point of its length form the inner parts or heart of the cable at another part of its length form the outside.

The object to be achieved is to construct some kind of stranding by which all portions of the cable are equally accessible from the dielectric, so that the energy arriving from the dielectric finds all parts of the mass of the cable, both surface and interior, equally accessible. In order to avoid external inductive disturbance, the proper form to give to a cable intended to convey rapidly intermittent or alternate currents is a couple of rather thin concentric tubes of copper well insulated from each other, and both insulated from the earth, of which one forms the *lead* and the other the *return*. By this device the metal will be most economically employed. An equivalent device would be a central core of stranded copper cable covered with insulation, and then plaited over with a sheath of other copper wires which form the return conductor.

In a further experiment, Lord Rayleigh (*loc. cit.*) examined the resistance of an iron wire of hard Swedish iron 10·03 metres long and 1·6 millimetre in diameter. In arbitrary units the resistance of the wire to steady currents was 10·4 units or ·51 ohm, and to currents of 1,050 complete alternations per second its resistance was 12·1 units, or ·595 ohm, which is an increase of about 20 per cent. In the case of a stouter wire, 18·34 metres long and 3·3 millimetres in diameter, the resistance to steady currents was 4·7 units, and the resistance to the interrupted currents of the above-mentioned frequency was 8·9 units, or nearly double. This illustrates the fact that for a given frequency of alternation the ratio in which the resistance is increased is greater the greater the diameter of the conductor, assuming it to be a round solid rod.

Lord Rayleigh found it more convenient in many researches to slightly alter the arrangement of the induction balance as described by Prof. Hughes, and to make it as follows (Fig. 36):—Two arms of a quadrilateral, R and S, consist of equal resistances of German-silver wire, wound double, so as to have negligible inductance. One arm, Q, consists of a coil having inductance and resistance greater than that of any conductor,

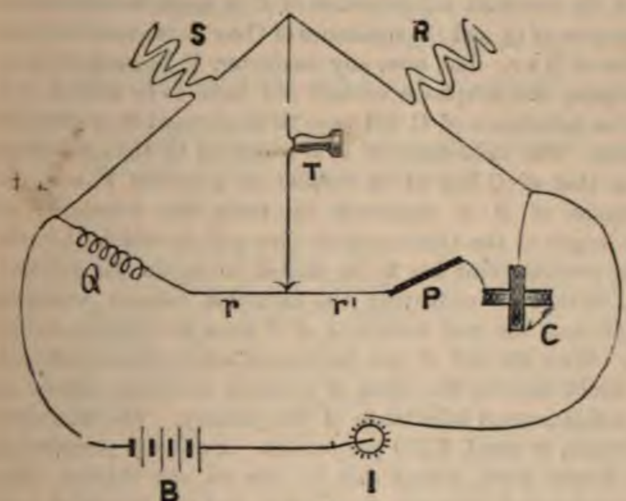

FIG. 36.

P, to be placed in the fourth arm. B and I are a battery and an interrupter, T is a telephone in the "bridge," and $r r^1$ is a German-silver wire of appropriate resistance, along which slides the contact of the bridge. The arm P includes a pair of coils joined in series, and which act upon each other by mutual induction, so that the resulting self-induction of the two coils in series can be varied within certain limits by turning one coil round within the other. For the resulting self-induction of such a pair of coils used in this manner may be regarded as made up of the component self-inductions of each coil taken separately and of twice the positive or negative mutual self-induction, depending upon which faces of the coils are presented to each other. It is possible, then, within certain limits to vary the inductance of the branch P C, and to vary also the resistance of the branches Q and P C by shifting the contact of the telephone along $r r^1$.

The condition for obtaining a true balance when the current is periodically interrupted is that the resistances and inductances of the branches Q and P C shall be separately equal. Suppose a balance has been obtained without the use of P, in which the resultant self-induction of C is made to balance the inductance of Q, and the resistance of $C + r'$ is made to be equal to that of $Q + r$. Let, now, any conductor, P, be inserted as in the figure, the telephone contact will have to be shifted, and *also* the inductance of C will have to be changed to re-obtain a balance. The inductance of P is measured by the amount by which that of C has to be *reduced* on inserting P, and the resistance of P is measured by twice the resistance of that length of the German-silver wire $r r^1$ by which the telephone contact point has to be shifted to regain the balance. This method of employing the induction balance separates out at once the real resistance of P from its effective induction. With the aid of this balance an interesting experiment was made, showing the effect of a closed secondary circuit on the resistance and inductance of the primary. The frequency was again, as usual, 1,050 per second. A coil was prepared of two copper wires, wound side by side on one bobbin. The diameter of each wire was about ·08in., and the length of each wire 318in. There were 20 (double) turns, so that the mean diameter of the coil, wound as compactly as possible, was about 5in., and the resistance of each wire was ·05 ohm.

The coefficient of mutual induction of the two wires was determined by comparison of the self-induction L of one wire with that of the two wires connected oppositely in series, viz., $(2 L - 2 M)$. In this way it appeared that

$$M = 43^{\circ} \cdot 1 = 43 \cdot 1 \times 1{,}553 \text{ centimetres.}$$

Observation showed that closing of the circuit of one wire reduced the self-induction of the other from 44°·4 to 3°·4. The resistance to steady currents was ·92 (arbitrary units). The resistance to the periodic currents was ·97 with the secondary circuit open, and 1·74 with the secondary circuit closed.

Hence, $L = 44{\cdot}4 \times 1{,}553$ centimetres, and
$R = {\cdot}97 \times {\cdot}0492 \times 10^9$ centimetres per second.

From Maxwell's formulæ, page 153, we get

$$\frac{p^2 M^2}{R^2 + p^2 M^2} = \frac{10^{17} \times 1{\cdot}951}{10^7 \times {\cdot}023 + 10^{17} \times 2{\cdot}071} = {\cdot}932.$$

According, then, to the formula $L^1 = L\,(1 - {\cdot}932)$,
or L^1 (decreased inductance) $= {\cdot}068\,L$,
$= {\cdot}068 \times 44{\cdot}4 = 3^{\circ}$.

And the observed value is 3°·4, which is in very tolerable agreement.

Again, the steady resistance with secondary open is ·92, and hence the resistance R^1 with secondary closed is

$$R^1 = 1{\cdot}932 \times {\cdot}92 = 1{\cdot}77\ ;$$

and observation gives the value 1·74. We see, then, that observations with this bridge confirm with a considerable degree of accuracy the deductions from the theory of simple periodic currents, that the closing of a secondary circuit increases the resistance and diminishes both the inductance and the impedance of an adjacent primary circuit.

From a practical point of view the most important difference between the conduction of steady electric currents and rapidly periodic currents is that of the *locale* of the current in the conductor and the consequent rise in the ohmic resistance of the conductor as a whole when employed with such periodic currents. Prof. Hughes called attention in 1883 to this great difference in the resistance of an electrical conductor if measured during the *variable* instead of the *stable* condition of the current.*

In experiments with his induction bridge Prof. Hughes was able to assure himself that the resistance of an iron telegraph wire of the usual size was more than three times greater for rapid periodic currents of about 100 per second than for steady

* Discussion on a Paper by Mr. W. H. Preece on "Electrical Conductors," *Proceedings* Inst. Civil Engineers, Vol. LXXV., 1883.

currents. The full elucidation of the propagation of currents in conductors under periodic electromotive force is not to be attempted without following out some very elaborate mathematical analysis. The subject has received its most complete treatment perhaps in the published writings of Mr. Oliver Heaviside which have appeared during many past years in the pages of *The Electrician*, and all that can be attempted here is to give a slight sketch of the views which are beginning to present themselves on this point. Consider a long level tank or canal full of liquid. There are, amongst others, two ways in which we might suppose this liquid to be set in motion. A paddle or the hand might be placed in the liquid, and by giving the liquid bodily a push it might be made to move forward; or we might suppose some body floating on the surface, such as a plank of wood, to be dragged along the surface. The friction between the plank and the layer of water beneath it would then cause the subjacent layer of liquid to move with the plank, and the motion of this layer would be gradually communicated to the other and deeper-lying layers by reason of the viscosity of the fluid. Or take the case of a basin containing water. The liquid might be set in rotation by stirring it with a paddle or the hand, but it might also be set in rotation by twisting the basin rapidly. In this last case the rotation of the basin would be communicated by friction to the water in contact with its sides, and then handed on from layer to layer of the water by internal fluid friction. Thus the twist or spin of the basin would be gradually propagated inwards from circumference to the centre. Imagine the whole mass of the liquid divided up into very thin concentric shells, like the coats of an onion. If the liquid were a perfect fluid there would be no friction between these layers, but since every liquid possesses some degree of viscosity or internal fluid friction, the sliding of one layer of fluid over another gradually causes the second layer to partake of the motion of the first. Hence, when the rotation of the basin commences the friction between its sides and the first layer of fluid starts that gradually in motion; this motion is then handed on to the second layer, and so forth, until the whole mass of the liquid possesses an equal angular velocity round the axis of rotation. The greater the fluid friction or viscosity the more rapid will be this equalisation of angular velocity of all parts of the fluid, and hence a rotating vessel full of tar would arrive at a stationary condition as regards such angular velocity sooner than one filled with a limpid liquid such as alcohol or ether. Just as the angular velocity diffuses inwards from the circumference to the centre in the case of such a revolving basin of liquid, so, according to modern views, does the current diffuse inwards from the circumference to the axis of the electric conductor. The student who has been accustomed to think of a current as produced in a conductor by a sort of push given to it *in* the conductor—such conception being based on a rough working hypothesis of a hydrodynamic nature—will perhaps have some difficulty in discarding this notion and realising that the current in a wire may perhaps be generated in it by an action taking place at all parts of the *surface of the wire* which gradually soaks or diffuses into the conductor out of the surrounding dielectric, but he will find that this new hypothesis serves to establish a mode of viewing the induction phenomena which makes various experimental results much more easily correlated. It was well demonstrated by the experiments of Prof. Hughes and others that a flat sheet or strip of metal has a less self-induction than a round wire of equal cross-sectional area. On the new hypothesis, this is explained by saying that the flat strip offers a greater absorption surface to the dielectric ; the current therefore soaks in more quickly to the centre and arrives at a uniform distribution over the cross section very soon—in other words, the variable state is sooner over, and we express this fact by saying that the self-induction is small. Again, if the electromotive force is oscillatory or rapidly periodic, we see at once that the current has not time to penetrate right into the core of the conductor before its sign or direction is reversed. It has hardly started on its journey inwards, soaking from surface to centre, before it is recalled; hence the flow of current when very rapidly periodic is confined to the surface of the conductor, the real or ohmic resistance is *increased*, and the self-induction is *diminished*.

Sir W. Thomson has shown (Bath British Association Meeting, 1888) that for alternate currents of frequency, equal to about 150 complete alternations per second, the depth to which the currents penetrate into the substance of the copper is about three millimetres, so that portions of the conductor beyond this distance from the surface are almost useless for conduction. The practical moral of this is that the proper form for a conductor for alternate currents is either a flat sheet of copper or a copper tube, in which, for the above frequency, the thickness of material is not more than one-quarter of an inch. To return to our illustration of the twisting basin of fluid. Suppose the action on the vessel consists in rapidly twisting it through a small angle, first one way and then the other, the liquid in the interior would be subjected to a strain which would consist in the various concentric layers of the liquid sliding backwards and forwards over each other. The interior of the liquid would be thrown into stationary waves, in which the nature of the wave motion consisted in each particle of water being displaced first one way and then another along an arc of a circle described on a horizontal plane, with its centre in the axis of rotation. The more rapid the motion the greater would be the rate of decrease in the amplitude of each wave in passing from the circumference to the centre of the vessel; in other words, for very rapid oscillations the bulk of the water in the centre of the basin would remain nearly at rest.

Every experiment as yet made on the self-induction or change of self-induction in conductors is consistent with the above hypothesis. It shows, for instance, why a conductor composed of thin insulated wires or thin insulated strips has a less self-induction than a solid conductor of equal cross-section. Prof. Hughes says*: — "We can reduce the self-induction of a current upon itself to a mere fraction of its previous force by simply separating the contiguous portions of a current from each other, the results proving that a comparatively small separation, such as is obtained by employing ribbon conductors in place of a wire of the same weight, reduces the self-induction 80 per cent. in iron and 85 per cent. in copper, and if we still divide the current by cutting the ribbon into several strips (separating the strips at least 1 centimetre from each other), then the combined but separated strips show a still greater reduction, being 94 per cent. in iron and 75 per cent. in copper."

* Inaugural Address, *Journal Soc. Tel. Eng.*, 1886.

These, and many other experiments of a similar sort, indicate that we may regard the inductance of a conductor as an effect which is due to the fact that the current takes *time* to penetrate into the conductor, and that a reduction of the time required to arrive at an equal current density in all parts of the conductor can be effected by any change of form which brings the inner parts of the conductor nearer the surface, or makes them more get-at-able from the dielectric. The better the conductor the slower is the rate of equalisation of current density over its cross-section—in other words, the less rapid is the rate of diffusion of the current inwards from centre to circumference; and the "time constant" of the circuit, or the time in which, under the operation of a constant electromotive force, the current will rise to a definite fraction of its maximum value, is a quantity proportional to the conductivity of the circuit, and to another factor (the formal inductance), which may be considered as expressing the accessibility of the conductor as regards geometrical form to the entrance of the current into it, and finally, in the case of magnetic conductors, to a quantity (the permeability) determined by the capacity of the conductor to utilise part of this incoming energy in producing magnetisation of its substance.

We are indebted to a Paper read before the Austrian Academy by Prof. Stefan for a simple and intelligible analogy helping the comprehension of the electrical distribution of current in a conductor. Imagine a cylinder or cylindrical wire heated throughout to a uniform temperature; let it be suddenly brought into a chamber where the temperature is higher. The outer layers of the cylinder will rise first in temperature, and gradually convey the heat to the successive interior layers. Precisely the same order of phenomena occurs if an E.M.F. is suddenly set up between the ends of the wire or cylinder. The current during the variable state passes first through the outer layers alone, and gradually penetrates the inner layers. When the external E.M.F. is suddenly removed the action of ceasing in the current resembles the *cooling* of the cylinder. The current ceases first, or, rather, most quickly, in the outer layers.

Now, let us imagine the cylinder transferred to and fro from a very hot place to a cool one. It is easy to see that waves of heat will pass in and out radially, and also that the condition at any instant will depend largely upon the rate of transference.

When the rate of motion is sufficiently slow the waves of heat passing any given point in the radius of the wire follow exactly with the periodic changes of position. The *amplitudes* of these variations have values which decrease from the surface inwards. When the rate of change is increased, the amplitude of the waves gets shorter and shorter, and at an infinite velocity of transference the wire would acquire an equable temperature throughout. In the electrical analogue the rate of transference corresponds to the inverse of the periodic time of an alternating current. The heat *conducting* power of the material corresponds to electrical *resistance*.

Prof. Stefan gives some numerical illustrations which are useful. If an alternating current have a frequency of 250 per second and is passed through an *iron* wire of 4 mm. diameter, the amplitude of the waves of current density is about twenty-five times as great upon the surface as at the axis of the wire. For double the number of vibrations per second the external amplitude becomes only six times as great. The difference of phase is one-third the duration of the vibration in the first case and one-half in the second. The latter statement implies that the external current is at a given moment actually in the reverse direction to the internal current.

For non-magnetic wires the difference is not nearly so marked, and it decreases as the specific resistance increases. For a copper wire of 4 mm. diameter, with a periodic time of one 500th second, the difference between the current density at the surface and at the centre is only 14 per cent. If, however, the copper wire be increased to 20 mm. diameter, then we should get the same difference as in the particular iron wire quoted.

It is obvious that this non-homogeneous distribution of current must increase dissipation of energy, which is, of course, proportionate in each transverse section to the square of the current strength at that spot. In the case of the iron wire quoted, the increase of resistance is 48 per cent. at the 250 per second frequency, and 100 per cent. at the higher speed. As the frequency of alternation is increased, the resultant self-induction of the circuit is lessened, but although the true resistance is increased, the impedance may be diminished on the whole.

§ **12. Induction Coils with Iron Cores.**—In the cases of mutual induction so far examined the problem has been treated under the assumption that the magnetic circuit had a constant permeability equal to unity; in other words, the magnetic circuits have been assumed to be air magnetic circuits. The inductance has been assumed as constant, and the induction, whether self or mutual, has been taken as proportional to the currents and to constant coefficients L, M, and N. In actual practice this is hardly ever the case. Induction coils, whether intended

FIG. 37.

"Choking Coil," or Circular Solenoid wound on a core of divided iron.

to form simple circuits of large self-induction, or to form mutually inductive circuits, have their magnetic circuits partly or wholly of iron, and accordingly the most important cases to examine are those in which iron cores are present, because these are the cases with which we have to do in practice. We shall then proceed to notice how the results previously obtained are modified by the presence of iron.

We take first the case of a single circuit forming a self-inductive coil or "choking" coil. Let Fig. 37 represent a circular sectioned ring of iron wound over closely with insulated wire. Let l stand for the mean perimeter of the ring or length of its circular axis. Let a be the area of cross-section of the ring.

Let there be N turns of wire on the ring having an ohmic resistance R. Let a simple periodic electromotive force be applied to the ends of the coil $x\,y$.

It is required to determine the current strength i at any instant in the circuit. We shall furthermore suppose that the iron is so divided parallel to the circular axis that no eddy currents are produced in the iron core, and neglect for the moment the effect of hysteresis and anything like magnetic lag. That is to say, we shall assume that the variations of the total induction B round the ring, or the total number of lines of force through the cross-section of the iron, varies immediately with the current and is in consonance with it as regards phase.

If b is the total induction at any instant, and i the current, and if B and I are the maximum values of these quantities, we can then write $b = \mathrm{B} \sin p\,t$, and $i = \mathrm{I} \sin p\,t$, p being, as usual, $2\,\pi$ times the frequency of the oscillations, or number of complete oscillations per second. Magnetisation is then assumed to follow in phase the magnetising force. We shall see afterwards how the results are modified by the introduction of hysteresis. The time is reckoned from the instant when the current or the magnetic induction is zero. Let e be at any instant the impressed electromotive force at the terminals $x\,y$. Then by the principles explained in Chap. III.

$$e = \mathrm{N}\frac{d\,b}{d\,t} + \mathrm{R}\,i \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

An equation expressing the fact that at any instant the impressed E.M.F. is equal to the sum of the effective electromotive force R i, and the inductive counter-electromotive force, or back E.M.F. $\mathrm{N}\frac{d\,b}{d\,t}$. The quantity $\mathrm{N}\frac{d\,b}{d\,t}$ is the expression of the fact that this back E.M.F. or counter-electromotive force of self-induction is equal to the rate of change of the quantity N b, or the total number of lines of force embraced by the circuit. Since $b = \mathrm{B} \sin p\,t$, we have

$$\mathrm{N}\frac{d\,b}{d\,t} = \mathrm{N}\,p\,\mathrm{B} \cos p\,t;$$

and since $i=\mathrm{I}\sin pt$ we have by substitution in (1)

$$e=\mathrm{N\,B}p\cos pt+\mathrm{R\,I}\sin pt.$$

By the lemma on page 132 we can write the above otherwise, as

$$e=\sqrt{(\mathrm{R}^2\mathrm{I}^2+\mathrm{N}^2p^2\mathrm{B}^2)}\sin(pt+\theta),$$

or, say, $e=\mathrm{E}\sin(pt+\theta)$,

where θ is such an angle that

$$\tan\theta=\frac{\mathrm{N}p\mathrm{B}}{\mathrm{R\,I}}.$$

Also by the general proposition that the line integral of magnetic force round any circuit is equal to 4π times the total current through the circuit, we have $4\pi\mathrm{N\,I}=l\,\mathrm{H}$, where H is the magnetic force in the core corresponding to the total induction B. Hence also $\mathrm{B}=\mu a\mathrm{H}$, where μ is the magnetic permeability corresponding to the total induction B, and a is the cross-section of the core. Accordingly

$$4\pi\mathrm{N\,I}a\mu=l\,\mathrm{B},$$

or,

$$\frac{\mathrm{B}}{\mathrm{I}}=\frac{4\pi\mathrm{N}a}{l}\mu.$$

Substituting this value of $\frac{\mathrm{B}}{\mathrm{I}}$ in the value for $\tan\theta$ we have

$$\tan\theta=\frac{4\pi\mathrm{N}^2\frac{a}{l}p}{\mathrm{R}}\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This angle θ is the angle by which the current lags behind the impressed electromotive force in phase, for when $pt=0$ we have

$$e=\mathrm{E}\sin\theta.$$

We see, then, that the tangent of the angle of lag is a quantity which depends upon the magnetic permeability. Two cases have to be considered. We know from the general form of the curve of magnetisation of closed iron circuits (*see* p. 65) that in the initial stage of magnetisation the magnetisation curve or curve for (B,H) rises upwards towards a "knee" or bend, and afterwards becomes more flat. The value of μ rises from a low initial value of about 100 towards a maximum of some 2,000—2,500 or more in the case of soft iron. The value $\mu = 100$ corresponds to very feeble magnetic force or induction, and the value $\mu = 2,000$ corresponds to an induction of something like 6,000—10,000 C.-G.-S. units, and to a magnetising force of from 3—4, or thereabouts, in C.-G.-S. units. It is obvious, therefore, that if the variation of current strength which takes place during a complete period or current wave is such that the magnetising force in the iron varies over considerable limits, the value of μ, and hence of tan θ, is changing from instant to instant. In the first place, let us simplify the problem by supposing that μ is a constant quantity, and does not vary much between the

Fig. 38.

limits of the magnetising forces produced by the total range of the current strength in its period. We can then easily represent graphically the change of current strength in the circuit corresponding to the variations of the impressed electromotive force, assuming this last quantity to have a simple periodic variation, or to vary according to a simple sine law.

Draw any line O E (*see* Fig. 38), and suppose it to represent in magnitude the maximum value of the periodic E.M.F. Let it revolve round its extremity O, and then its projections from instant to instant on any line O y represent the magnitude of this periodic E.M.F. at those instants. Next, let us suppose the constant quantities R, N, a, l, p, and μ given, being the electrical, geometrical, and magnetic data of the coil. Calculate the quantity

$$\tan\theta = \frac{4\pi\mu N^2 a p}{R l},$$

and set off O C, making with O E the angle E O C equal to θ. On O E describe a semicircle E C O, and from the point C where O C intersects this curve draw C E, joining C and E. Then on the same scale on which O E represents the maximum impressed E.M.F., O C represents the maximum effective E.M.F., viz., R I acting in the circuit, and the line C E represents the maximum inductive E.M.F. or N p B. For the sum of the projections of E C and O C on O y is equal to the projection of O E on O y, and E C is to O C in the ratio of N p B to R I by construction, and the phase of N p B is in quadrature with that of R I. Hence, taking one Rth part of O C, say O G, and projecting O G on O y we have O i, and O i represents the magnitude of the current i, corresponding to the instant when the impressed E.M.F. has a value e, represented by O a; and O G and O E represent the relative maximum magnitudes of these quantities and their relative phase, on the assumption that a unit of length stands for a unit of current, of resistance, and of electromotive force. In actual practice, if the quantities μ, N, and p are considerable numbers, it will be found that tan θ is also a very large number, or θ is nearly 90°; in other words, such a coil impresses on the current a lag of nearly 90°, and it is easy to see that in this case the value of the ratio of O a to O i is always a great one; that is to say, the action of a sinoidal or simple periodic E.M.F. results only in the production of a very small average current relatively to that which would be produced if the mere ohmic resistance were the only quantity taken into account in calculating the current. Such a coil is called, for this reason, a "choking coil," because it throttles or chokes the current. We have, however, shown that the rate of energy dissipation in such a circuit, or the power taken up, is represented by half the product of the maximum of the impressed E.M.F. and current and the cosine of the angle of the lag, or to

$$\frac{1}{2} \mathrm{E\,I} \cos\theta.$$

Hence, if θ is large, cosine θ is very small, and the result is that the power absorbed in the coil is also small.

Summing up the results we may say that if a simple periodic E.M.F. is applied at the terminals x y of a choking coil, and assuming a constant magnetic permeability of the core, the result is to produce a simple periodic current, lagging behind the E.M.F. in phase by a constant angle.

Turn next to consider the case when μ is not constant, or when the variation of current strength during the current wave is so great that μ cannot be considered as constant during the various stages of the phase. In this case the angle of lag of current behind impressed electromotive force depends on the value of the current at that instant. For the lag depends on μ, and μ depends on the magnetic induction, and hence on i.

FIG. 39.

If i is very large at its maximum value then the magnetising force will be large also, but corresponding to this the ratio of induction to magnetic force *may* be small, and hence the lag small also. Accordingly, when the current varies over such a range during its period that the magnetic induction is carried up well over the "knee" of the magnetisation curve, the value of μ will *pulsate* with the current, though not simply proportionately, and the angle of current lag will pulsate regularly, being small when the current is at its maximum, and large when the current produces that induction corresponding to which μ is a maximum. Hence it follows that even if the impressed electromotive force varies according to a simple sine law the current will *not* vary according to the same law, but will be greater at and about its maximum value than it would be if the current variation depended on a constant lag equal to the average lag. In fact, the form of current curve might be represented in a rough manner graphically by some such kind of curve as 1 1 in Fig. 39, the E.M.F. curve being marked E E, and a true sine curve current for comparison being represented by the dotted line.

Still, neglecting for the moment any effect of hysteresis, we find a starting point for the process of determining the current corresponding to any form of impressed electromotive force on the following principles.

At any instant, as before, the impressed electromotive force e balances the effective electromotive force $R\,i$, and the inductive electromotive force $N\frac{d\,b}{d\,t}$;

or,
$$e = R\,i + N\frac{d\,b}{d\,t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But
$$\frac{d(N\,b)}{d\,t} = \frac{d\,(N\,b)}{d\,i}\cdot\frac{d\,i}{d\,t}.$$

Now $\frac{d\,(N\,b)}{d\,i}$ is the rate of change of the total induction through the circuit with the current, and is the quantity by which, at that instant, the time rate of change of the current or $\frac{d\,i}{d\,t}$ has to be multiplied to obtain the counter-electromotive force of induction; in other words $\frac{d\,(N\,b)}{d\,i}$ is the coefficient of self-induction, or the inductance of the circuit *at* that instant. It has been pointed out on page 62 that the inductance of a circuit wrapped round an iron core is a function of the magnetisation of that core, and can be obtained for any given value of that magnetisation from the magnetisation curve of the iron circuit. In the present instance the quantity $\frac{d\,(N\,b)}{d\,i}$ is that which was called L_1 on page 59. Substituting these values, the fundamental equation becomes

$$e = R\,i + L_1\frac{d\,i}{d\,t}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Let $\frac{e}{R}$ be called i_0. It is the current which would exist in the circuit at that instant if inductance were annulled. Also write T for $\frac{L_1}{R}$, T being the time constant at that instant when the current is i. We have then by further substitution in the equation (4),

$$i + T\frac{di}{dt} = i_0,$$

or

$$\frac{di}{dt} = \frac{i_0 - i}{T},$$

in which, be it remembered, T stands for $\frac{1}{R}\frac{d(Nb)}{di}$.

Furthermore, we have to bear in mind that

$$4\pi N i = l H,$$

so that

$$\frac{1}{R}\frac{d(Nb)}{di} = \frac{N}{R}\frac{db}{di} = \frac{N}{R}\frac{db}{dH}\frac{dH}{di},$$

or

$$\frac{1}{R}\frac{d(Nb)}{di} = \frac{4\pi N^2}{Rl}\frac{db}{dH} = T.$$

In order, then, to find T corresponding to any value of i, we have to know the value of $\frac{db}{dH}$, or the rate at which magnetic induction is changing with respect to magnetic force at that instant. This is at once given from the magnetisation curve of the iron, for $\frac{db}{dH}$ is the tangent of the angle of the slope of the geometrical tangent drawn to the magnetisation curve at that point which corresponds to the given value of i.

Suppose any value of i given, we can at once calculate the value of H, for it is $\frac{4\pi N}{l}\,i$.

Look out on the magnetisation curve the abscissa equal to H. Let it be O H (*see* Fig. 40). Draw P H an ordinate at that point, then the magnitude of P H gives us b, and if we draw at P a tangent P M the tangent of the angle P M H is the value of $\frac{db}{dH}$, and multiplication of this by $\frac{4\pi N^2}{Rl}$ gives us the value of T corresponding to the given value of i. It is thus possible to draw a curve showing the variation of T with i. We can here follow with advantage a construction which has been given by Mr. W. E. Sumpner (*Phil. Mag.*, June, 1887, p. 470).

Let T_1 T_2 (Fig. 41) represent a time constant curve plotted so that the vertical ordinates parallel to O y represent values of i, or current strengths, and the abscissæ parallel to O Z represent values of T or $\frac{L_1}{R}$ calculated as described. This curve must be obtained for the particular iron circuit in question, because variations of quality of iron, and the nature of the subdivision

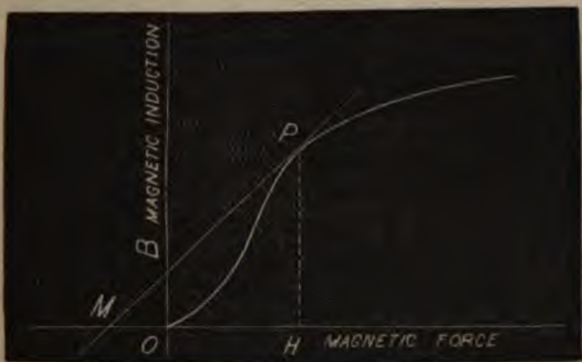

FIG. 40.

of the iron, will affect its form. It will be found convenient thus to plot T in the negative direction. The time ratio T will be in seconds if L_1 is in secohms and R in ohms. On the right hand side of O Y plot another curve, C_0, representing in vertical ordinates the value of $\frac{e}{R}$ for the different values of e corresponding to various times plotted as abscissæ along O X. Suppose P_1 be a given initial point on the real current curve connecting the values of i with time. Project P_1 parallel to O Z and O Y on to the curves of C_0 and T. Project Q_1 on to O Y, and join T_1 R_1. From P_1 draw a *short* line parallel to T_1 R_1, and then if P_2 is a point very near to P_1, P_2 will be the next point on the current curve. It is thus possible to obtain a series of points belonging to the curve of i. It is obvious, since the length $P_1 Q_1$ represents $(i_0 - i)$, and since $S_1 T_1$ represents T, that the tangent of the angle $R_1 S_1 T_1$ represents $\frac{i_0 - i}{T}$, and since the element of the curve $P_1 P_2$ is drawn parallel to $T_1 R_1$ that this element must belong to a curve

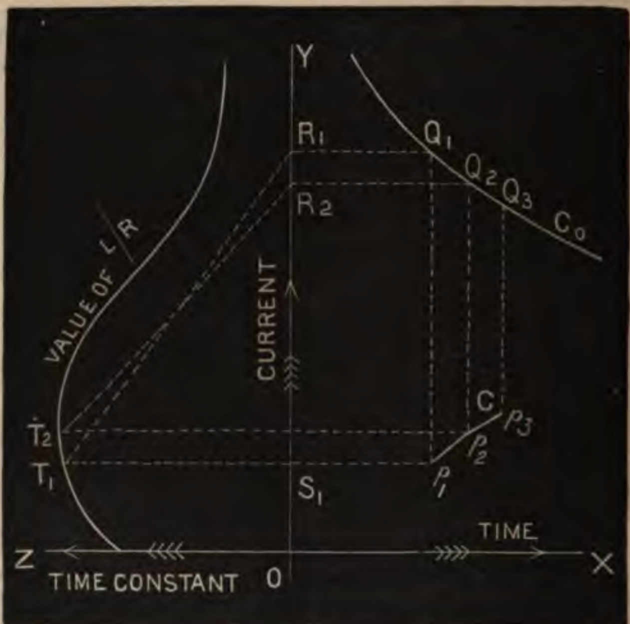

Fig. 41.

for which the slope of the curve at P_1, or the $\frac{di}{dt}$, is equal to $\frac{i_0 - i}{T}$. Hence these points P_1, P_2, etc., fulfil the conditions for being points on the curve representing the change of actual currents with time. It will probably facilitate a comprehension of this geometrical construction if we apply it to drawing the form of the curve which delineates the gradual rise of current strength in an electro-magnet when a steady electromotive force is applied to the ends of the magnetising coil (*see* Fig. 42).

In this case the curve i_0 becomes a straight line, for if inductance were absent the current would be produced instantaneously at its full value $\frac{E}{R}$, where E is the impressed electromotive force and R is the ohmic resistance of the coil.

Let O X, O Y be axes parallel to which are measured *time* and

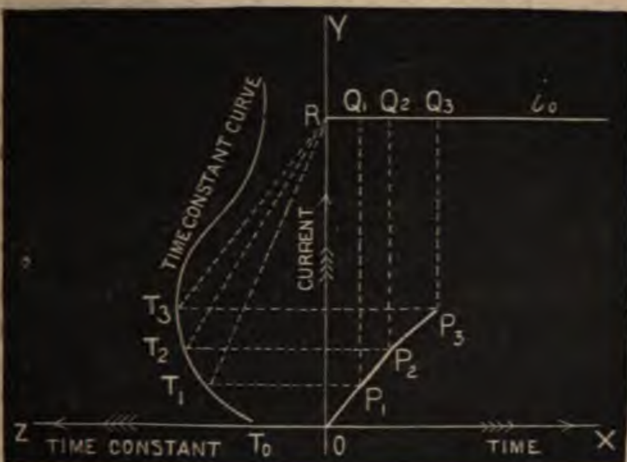

Fig. 42.

current strength, and along O Z[1] in the negative direction let the time constant values be plotted. Let the curve T_1 T_2 T_3 be first drawn indicating the change of time constant L ÷ R obtained from the magnetisation curve, as drawn for the particular magnet considered. Then let a straight line i_0 be drawn parallel to O X, and at a distance E ÷ R from it. Beginning at O, join R and T_0, the point where the time constant curve cuts the axes of X, and draw a short line O P_1 parallel to T_0 R. Take the point P_1 very near O and project P_1 into i_0 and on to the time constant curve at T_1. Join R T_1, and draw a short line P_1 P_2 parallel to T_1 R, and proceed again to determine an adjacent point P_2, and so on. The curve freely drawn through O, P_1, P_2, &c., is a curve indicating the rise of current with time. This curve gradually rises up to alignment with the line i_0. The form of this current curve will depend on the form of the time constant curve. If the self-induction of the circuit is constant, or nearly so, the slope of the current curve will be great at first, but continually diminish as time goes on; if, however, the self-induction be very variable the result may be quite different, and depend on the final value i_0. Suppose the self-induction to be great at first and then to diminish, as shown in Fig. 22, page 63, unless i_0 is very small there will be points of inflexion on the current curve. The current will increase very rapidly at first, slower afterwards, and then more rapidly, and finally attain its maximum very slowly. If the maximum current is such as to magnetise the iron far beyond saturation, this effect may be very marked, and the time taken for the current to rise to a small fraction of its final may exceed the time taken to rise through the remainder. Mr. Sumpner remarks that this phenomenon has been observed when the current from a secondary battery was used to excite the field-magnets of a dynamo, with an Ayrton and Perry dead-beat ammeter in circuit. The needle was noticed to move slowly at first and then with great rapidity through the large portion of its ultimate deflection.

When the impressed electromotive force is alternating and sinoidal, or simply periodic, the construction shows that the form of current curve is not a sine curve, but may run up into a sharp peak; in fact, if the current rises up to such a value that the magnetisation of the iron is carried away over the "knee" of the magnetising curve, the coefficient of self-induction then falls down to a very small value relatively to its value when the core is about half saturated. The result is that the resultant current has a much greater actual instantaneous value than that which it would have if a constant inductance equal to the maximum value were maintained. The practical result of this is that a choking coil ceases to choke if the impressed E.M.F. has a value sufficiently large to more than saturate the iron. Hence the throttling or choking or impeding power of an electro-magnet may be broken down under these circumstances. This action has been noted by Mr. Kapp (see *The Electrician*, Vol. XVIII., p. 525).*

Since a closed iron magnetic circuit "saturates," in the popular sense of the term, for a lower magnetising force than an open iron air circuit, it would seem that this reduction of the throttling power could be deferred by making the magnetic iron circuit not quite closed—in other words, leaving an air gap in it. We have now to consider how the form of the current curve is affected by hysteresis. In this case, when the electromotive force is alternating, there are two values of the induction, and hence of the coefficient of self-induction, for each value of the magnetising force or current, depending on whether the current is increasing or diminishing. The curve T, or the time-constant curve, will therefore be a double-branched curve, one branch of which belongs to the ascending and one to the descending magnetisation. It is easy to see, therefore, that the resulting current curve will have a want of symmetry in its ascending and descending sides.

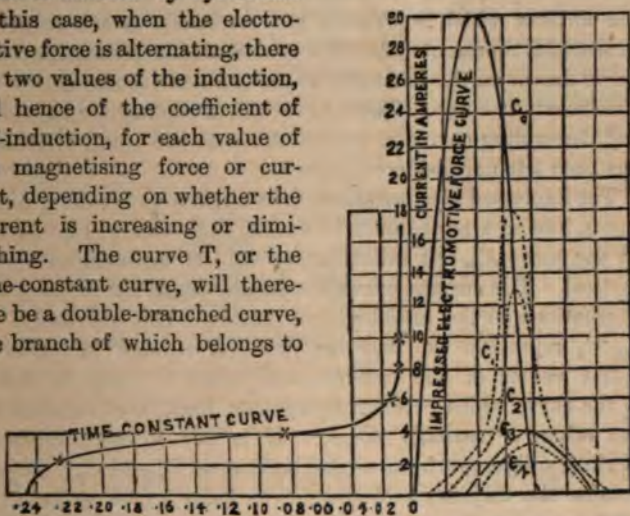

FIG. 43.
Electromotive Force and Current Curves of Transformer.

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* Mr. G. Kapp published in *Industries* of April 8, 1887, *et seq.*, an admirably-written series of articles on "Induction Coils Graphically Treated," in which the geometrical method previously employed by Mr. Blakesley and others was made use of to elucidate the chief phenomena of self and mutual induction. These articles were reprinted in *The Electrician*, Vol. XVIII., 1887, pp. 504, 525, and 568. They are recommended to the practical reader.

Even although the curve for E and for i_0 be simple sine curves, the resulting actual current curve may exhibit changes of curvature and want of symmetry equivalent to the superposition of ripples upon a simple periodic curve.

In Fig. 43 is shown an example of a current curve worked out by experiments made on a Kapp and Snell transformer. These experiments were made by Dr. Sumpner, at the Central Technical Institution, London. The resistance of the circuit was for simplicity in the diagrams taken as one ohm, so that the number which represents the coefficient of self-induction in secohms represents also the time-constant in seconds. The curve drawn on the left hand of the central vertical line is the time-constant curve. This curve differs slightly for ascending and descending currents, but in this case the small hysteresis has been left unconsidered.

The impressed electromotive force is taken as following a sine curve, having a period of 0·16 second, and this is represented by the curve C_0, the tallest curve on the right hand of the vertical. The current curve obtained depends for the first few alternations on the initial circumstances. The curves C_1, C_2, C_3, C_4 represent the first half wave of current for different initial values of the current. Of these, C_3 may be taken as the curve which periodically repeats itself, and to which all the others eventually come. It is not in appearance so markedly different from a sine curve as C_1 and C_2. This is because the impressed electromotive force is not sufficient to produce a current capable of magnetising the iron beyond its point of saturation at which the value of the inductance begins to diminish. Otherwise, the current curve would be characterised by having sharp peaks in it.

§ 13. **Transformers with Iron Cores.**—A pair of mutually inductive circuits so arranged that the coefficient of mutual induction is considerable is called in usual language an induction coil. In nearly all cases the magnetic circuit is partly or wholly of iron. In virtue of the fact that such a device merely translates the form of electric energy, or can be made so to do, in changing a current of high electromotive force into one of low, or *vice versâ*, it has in its practical applications with periodic currents been called a *Transformer*, or frequently also a *Converter*. The general problem of the induction coil with a magnetic circuit of constant permeability has already been treated on the assumption that the constants called the coefficients of induction are known and numerically given quantities. As, however, the presence of iron renders these coefficients variable and dependent on the induction it is preferable to handle the problem of transformers with iron cores in a different manner. We shall first follow the method of investigation employed by Mr. Kapp (*loc. cit.*),* altering, however, the treatment in several details. If two electric circuits are linked with a magnetic circuit of variable permeability forming a chain of three links (*see* Fig. 44), and if one of the electric circuits is traversed by a steadily periodic electric current, then we find

FIG. 44.

the following state of things present in such a linked magnetic and electric chain :—

1. A wave of periodic impressed electromotive force acting on the primary circuit, which will be assumed to have a simple periodic form, or to be a sinoidal function of the time.

2. A periodic primary current taking place in the primary circuit, of the same periodic time as the impressed E.M.F., but not necessarily sinoidal in form, and lagging behind the E.M.F. in phase.

3. A wave of counter electromotive force in the primary circuit due to inductive action, and not coinciding in phase either with the impressed E.M.F. or with the primary current.

* *The Electrician*, Vol. XVIII., p. 568, May, 1887. See also Mr. G. Kapp's Paper on "Alternate-Current Transformers," *Journal of the Society of Telegraph-Engineers*, Vol. XVII., Feb. 9th, 1888, p. 96.

4. A wave of magnetisation in the core lagging behind the primary current by something less than 90° of phase.

5. A wave of impressed electromotive force in the secondary circuit due to and measured by the rate of change of magnetic induction in the core, and hence lagging 90° or rather more in phase behind the magnetisation wave.

6. A wave of secondary current lagging behind the secondary E.M.F. in phase unless the secondary circuit consists of comparatively few turns of conductor, and is connected with an external circuit of practically no inductance, as is the case when the external circuit consists of incandescent lamps in parallel.

The problem then generally is to find the relation between these various currents, magnetisations, and electromotive forces as regards magnitude and phase, given certain dimensions and determinable constants.

We shall, as before, neglect, in the first place, magnetic hysteresis and suppose that the magnetic permeability is nearly constant, or, which comes to the same thing, suppose that the magnetic forces brought to bear on the core are not sufficient to quite saturate it or carry it beyond magnetic saturation in the ordinary sense of the term.

The general statement of the physical actions going on is that the variation of the primary current produces a variation of magnetic induction in the core, and this again in turn acts to produce an induced electromotive force, both in the secondary circuit and in the primary, the last being called the primary self-inductive electromotive force. Hence we can picture to ourselves the waxing and waning primary current to be followed by a rise and fall of the number of lines of force running in endless loops round the core, and this again by a pulsation of secondary current in the secondary circuit. At any instant the total induction in the core or total number of lines of induction linked with both circuits depends upon the resultant magnetising force to which the core is subjected, and this again depends upon the resultant ampere-turns acting to produce magnetising force. If we suppose the core to be of such a shape (say circular) that the magnetising force is everywhere the same in its interior, and if we suppose it, furthermore, to have a uniform section, then the line integral of the magnetic force round the core will be obtained by taking the product of this resultant magnetising force and the mean length of path of each line of induction. By the general theorem that 4π times the total flux of current *through* the core is equal to this line integral, we have a starting point for an investigation of the relative magnitudes of the currents, electromotive forces, and angles of lag.

Consider the case of a transformer (Fig. 45) having a circular ring-shaped iron core. Let the mean length of a line of induction round the core, represented by the fine dotted circular line by l. Let there be two intertwined circuits, Pr. and Sec., having ohmic resistances R_1 and R_2, and making N_1 and N_2

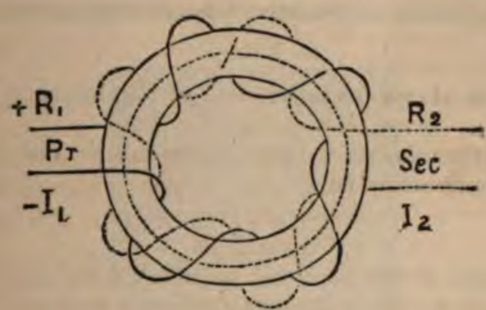

FIG. 45.

turns respectively round the core. Grant that the circuits have no sensible self-induction outside the coils coiled on the core, and that the secondary resistance R_2 includes that of any lamps or other apparatus through which a secondary periodic current is being sent which is generated by the periodic current produced in the primary. Let I_1 and I_2 be the maximum values during their respective periods of these currents, and let i_1 and i_2 be the values at any and the same instant of these currents. Also, suppose E to be the maximum value of the sinoidal impressed electromotive force and e its instantaneous value at any other moment.

We will furthermore take it that b stands for the value of the total induction through the core at any time t, and that B is the maximum value of b. If the uniform cross-sectional area of the core is a, then $\frac{b}{a}$ is *the induction* at the time t, and if μ is the permeability coresponding to this induction, then $\frac{b}{\mu a}$ is the magnetising force at that instant.

By our fundamental principle, then,

$$\frac{l b}{\mu a} = 4 \pi (N_1 i_1 + N_2 i_2) \quad . \quad . \quad . \quad . \quad (1)$$

or the line integral of the magnetic force round the core is equal to 4π times the total flux of current through the core. In the absence of anything like hysteresis, it may be taken that the induction follows the simple periodic law of the primary impressed electromotive force, and that, therefore,

$$b = B \sin p t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

p being as always $2 \pi n$, where n is the frequency of the oscillations.

If e_2 is the value at the same instant of the secondary impressed electromotive force,

$$e_2 = N_2 \frac{d b}{d t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for the value of this impressed secondary E.M.F. is given by the value of the rate of change of the total induction.

Hence, from (2) and (3),

$$e_2 = N_2 p B \cos p t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$i_2 = \frac{N_2 p B}{R_2} \cos p t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But if I_2 is the maximum value of i_2 during the period,

$$N_2 I_2 = \frac{N_2^2 p B}{R_2},$$

and

$$N_2 i_2 = N_2 I_2 \cos p t.$$

Hence, substituting the value of $N_2 i_2$ in (1) we have, since

$$N_1 i_1 = \frac{l}{4 \pi \mu a} b - N_2 i_2,$$

$$N_1 i_1 = \frac{l}{4 \pi \mu a} B \sin p t - N_2 I_2 \cos p t \quad . \quad . \quad (6)$$

Since $N_1 i_1$ must be a periodic function, and has a maximum value $N_1 I_1$, we can write it

$$N_1 i_1 = N_1 I_1 \sin (p t - \phi) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and by the theorem on p. 132 we see that ϕ is an angle such that

$$\tan \phi = \frac{N_2 I_2}{\frac{l}{4 \pi \mu a} B} = \frac{4 \pi \mu a N_2}{l} \frac{I_2}{B} \quad . \quad . \quad . \quad (8)$$

and that
$$N_1^2 I_1^2 = N_2^2 I_2^2 + \left(\frac{l B}{4 \pi \mu a}\right)^2.$$

We can then represent the relation of the primary ampere-turns to the secondary ampere-turns geometrically, thus:—Draw a right-angled triangle, O A B (Fig. 46)—

FIG. 46.

$A B = N_2 I_2$ = Secondary ampere-turns.
$A O = N_1 I_1$ = Primary ampere-turns.
$B O = \sqrt{N_1^2 I_1^2 - N_2^2 I_2^2}$ = Resultant ampere-turns.

Take O A to represent on any scale $N_1 I_1$ the primary ampere-turns, and the perpendicular A B to represent to the same scale the secondary ampere-turns. Then O B represents to the same scale the resultant ampere-turns, and represents the quantity $\frac{l}{4 \pi \mu a} B$; and the angle A O B represents the angle ϕ.

Suppose that in such a transformer we have measured the maximum values of the primary and secondary currents, obtainable, of course, from the mean square values as given by an electro-dynamometer; we proceed to construct a diagram giving the relative phases of the currents and the value of the electromotive force which must be impressed on the ends of the primary circuit to produce these currents.

Draw a circle whose radius O A numerically represents the primary ampere-turns $N_1 I_1$, and in this diagram let unit of length stand for a volt, an ohm, an ampere, and a turn. Draw

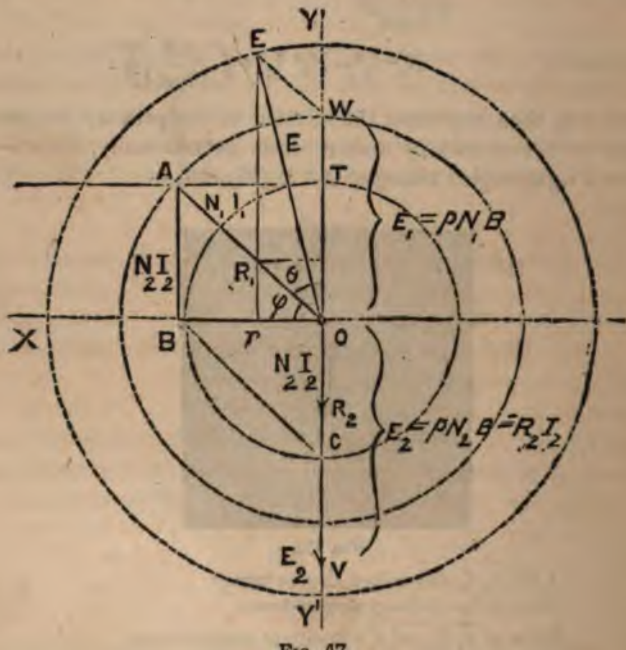

FIG. 47.

a concentric circle whose radius O C numerically represents the secondary ampere-turns $N_2 I_2$. Draw axes O X, O Y at right angles through the centre. Let A T be drawn tangent to the inner circle and parallel to O X, and at the point A where it cuts the second circle draw the radius O A. Let fall the perpendicular A B on O X and join B C.

Then O A B C is a parallelogram, of which the side O A represents the primary maximum ampere-turns, the side A B = O C represents the secondary ampere-turns, and the diagonal O B represents the resultant ampere-turns, and also represents the quantity $\frac{l\,B}{4\pi\,\mu\,a}$, or is proportional to the maximum of the core induction. The angle A O B represents the angle ϕ, by which the maximum primary current ~~lags behind~~ the induction or magnetisation of the core. Hence, by taking one N_1th of O A, and one N_2th of O C, we get radii which represents the proportionate magnitudes of the maxima of the primary and secondary currents, and their relative phase angles differ by an angle $90^\circ + \phi$. Also O B is proportional to the total induction B through the core, and this induction is in quadrature with the secondary current.

For the secondary current is in consonance as regards phase with the secondary impressed electromotive force on the supposition that there is no external inductance in the secondary circuit, and no lag of magnetic induction behind magnetising force, and hence, as the secondary inductive electromotive force is in magnitude proportional to the rate of change of the induction we see that both it and the secondary current are 90° in phase behind the magnetisation. This is also indicated by the trigonometrical functions in the equations (2) and (5). Accordingly, if we suppose the lines O A, O B, O C to revolve round O in fixed relative positions their projections at any instant on any diameter through O give us lines proportional in magnitude to the primary current, the magnetisation, and the secondary current, and the magnitudes of O A, O B, O C are in the same way proportional to the maxima of these quantities. Next mark off on O A a distance O R_1 such that OR_1 is to O A as R_1 is to N_1, then O R_1 represents $R_1\,I_1$, or the maximum effective electromotive force acting in the primary. Through R_1 draw the straight line $r\,R_1\,E$ parallel to O Y.

Now the variation of the induction induces in the secondary and primary circuits electromotive forces of induction whose maximum magnitudes we shall denote by E_2 and E_1, and these are numerically equal respectively to the maximum of the rate of change of the total induction, or $E_1 = p\,N_1\,B$

and $$E_2 = p\,N_2\,B$$

E_2 is in consonance as observed above with the secondary current, and moreover, E_2 is equal numerically to $R_2\,I_2$. E_1 is in phase exactly in consonance with E_2, but it acts in opposition to it.

Hence, to represent on the diagram E_1 and E_2, we must mark off a length O V on O Y^1, equal to $R_2 I_2$, and since

$$E_1 = p N_1 B,$$

and

$$E_2 = p N_2 B = R_2 I_2,$$

it follows that

$$E_1 = \frac{N_1 R_2 I_2}{N_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

So that O W can be set off along O Y to represent E_1, and we have

$$O W = E_1 = \frac{N_1}{N_2} R_2 I_2,$$

and

$$O V = E_2 = R_2 I_2,$$

as the values of the inductive electromotive forces set up in the primary and secondary circuits respectively.

Next we note that since O R_1 represents the effective electromotive force in the primary, and O W the self-inductive electromotive force, the impressed electromotive force O E is obtained by taking the resultant O E of the parallelogram described on O R_1 and O W as adjacent sides. For then since O E, O R_1 and O W revolve round and represent the maximum values of the impressed, effective and inductive electromotive forces, we shall always have

$$e = e_1 + e_r,$$

where e, e_1 and e_r stand for the simultaneous values at any instant of E, E_1 and $R_1 I_1$. Accordingly on the predetermined scale O E represents the maximum of the impressed electromotive force, and we see that it is ahead of the primary current by an angle θ.

From the geometry of the figure, since E R_1 is parallel to O Y, we see that

$$O r = E \cos (\phi + \theta),$$

and

$$O r = O R_1 \cos \phi$$

$$= R_1 I_1 \cos \phi.$$

Therefore

$$E \cos (\phi + \theta) = R_1 I_1 \cos \phi \quad . \quad . \quad . \quad (10)$$

Also, we see that $\sin \phi = \frac{N_2 I_2}{N_1 I_1}$.

Hence, sin ϕ, and therefore cos ϕ, are determined in terms of the primary and secondary currents and turns; and also the angles ϕ and $\phi + \theta$ are known in terms of N_1, R_1, I_1, and N_2, R_2, I_2. Hence, from the above observed and measured quantities we can determine the angle θ by which the primary current lags behind the primary impressed electromotive force, and also the angle ϕ by which the magnetic induction, or magnetisation of the core, lags behind the primary current, and, as a consequence, the angle $\phi + 90$, by which the primary and secondary currents differ in phase; maximum magnitudes being always understood. We can then proceed to determine, from the geometry of Fig. 48, two other equations connecting the magnitude of the impressed electromotive force and those of the currents.

Experiments, to which we shall allude presently, show that in transformers with complete magnetic iron circuits the angle $\phi + \theta$ is very nearly 90deg., and hence that E and E_1 are very nearly in consonance in phase. Hence it follows that when E and E_1 have their maximum values, we may, with very small error, take their difference, or $E - E_1$, as equal to the effective electromotive force in the primary circuit.

If at this instant the primary current has a value i_1, then the effective electromotive force in the primary circuit is $R_1\ i_1$, and we have under these conditions

$$E - E_1 = R_1 i_1.$$

Moreover, the phase difference of E and the magnetisation will be very nearly 90deg. Hence, since the value of i_1 at the instant when E has its maximum is $I_1 \cos \theta$, and since $\cos \theta = \sin \phi = \frac{N_2 I_2}{N_1 I_1}$, we find that $E = R_1 I_1 \sin \phi + E_1$ . . . . (11)

But by equation (9), page 276, we found that

$$E_1 = \frac{N_1}{N_2} R_2 I_2.$$

Substituting the values of sin ϕ and E_1 in (11) we have

$$E = R_1 I_1 \frac{N_2 I_2}{N_1 I_1} + \frac{N_1}{N_2} R_2 I_2,$$

or,
$$I_2 = \frac{E\frac{N_2}{N_1}}{R_1\left(\frac{N_2}{N_1}\right)^2 + R_2} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which may be written better

$$I_2 = \frac{E}{R_1\frac{N_2}{N_1} + R_2\frac{N_1}{N_2}} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

an equation which gives us a value for the secondary current in terms of the turns, resistances, and impressed electromotive force.

FIG. 48.

If the ratio of $\frac{N_2}{N_1}$ is a small fraction, say $\frac{1}{100}$, then the term $R_1\left(\frac{N_2}{N_1}\right)^2$ may be negligible in comparison with R_2, and in this case we see that the transformer transforms down the electromotive force in the ratio of the primary and secondary turns, and an approximate value of the secondary current will be given by dividing the quantity $E\,\frac{N_2}{N_1}$, or the sceondary impressed electromotive force, by the secondary ohmic resistance R_2.

Again, taking the same figure, we see that the parallelogram $O W E R_1$ is one whose sides $O R_1$, $O W$, represent respectively the self-inductive electromotive force E_1 in the primary and the effective electromotive force $R_1 I_1$, and that the diagonal represents the impressed electromotive force E.

Since $$(O E)^2 = (O r)^2 + (E r)^2,$$
and since $$O r = O R_1 \cos \phi$$
and $$E r = O W + R_1 r$$
$$= O W + O R_1 \sin \phi,$$
we have $$(O E)^2 = (O R_1)^2 + (O W)^2 + 2 O W, O R_1 \sin \phi,$$
but $O E$ represents E, the impressed electromotive force, and $O R_1 = R_1 I_1$ and $O W = E_1 = \frac{N_1}{N_2} R_2 I_2$, the effective and inductive electromotive forces respectively in the primary. Hence

$$E^2 = R_1^2 I_1^2 + E_1^2 + 2 R_1 I_1 E_1 \sin \phi.$$

Substituting the values of E_1 and $\sin \phi$, we reach

$$E^2 = R_1^2 I_1^2 + \left[\frac{N_1^2}{N_2^2} R_2^2 + 2 R_1 R_2\right] I_2^2 \quad . \quad (14)$$

an equation connecting the values of the primary and secondary currents and the impressed electromotive force.

Hence, given E and I_2 we can calculate I_1, or given I_1 and I_2 we can calculate E.

These equations (13) and (14) furnish the practical solution of the problem of the transformer.

Furthermore, we have the means of estimating the *efficiency* of transformation. If W stand for the true watts given to the transformer, we know that

$$W = \tfrac{1}{2} E I_1 \cos \theta,$$

and if w stand for the total watts expended in the secondary circuit,

$$w = \tfrac{1}{2} R_2 I_2^2.$$

Hence, the efficiency of transformation ϵ being equal to the ratio $\frac{w}{W}$, is given by

$$\epsilon = \frac{w}{W} = \frac{\frac{1}{2} R_2 I_2^2}{\frac{1}{2} E I_1 \cos \theta},$$

but under the circumstances considered

$$\cos\theta = \sin\phi \text{ and } \sin\phi = \frac{N_2 I_2}{N_1 I}.$$

Hence, $$W = \tfrac{1}{2} E \frac{N_2}{N_1} I_2.$$

Substituting the value of E in terms of I_2 from (13) we have

$$W = \tfrac{1}{2} \frac{N_2}{N_1} \left(\frac{N_2}{N_1} R_1 + \frac{N_1}{N_2} R_2 \right) I_2^2,$$

or, $$\epsilon = \frac{R_2}{\frac{N_2}{N_1} \left(\frac{N_2}{N_1} R_1 + \frac{N_1}{N_2} R_2 \right)};$$

therefore, $$\epsilon = \frac{1}{1 + \left(\frac{N_2}{N_1} \right)^2 \frac{R_1}{R_2}} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In all the foregoing equations the symbols E, E_1, E_2, I_1, and I_2 stand for the maximum values of the quantities during the period. If, however, the variation may be taken as sinoidal, or approximately so, the multiplication of each by the factor $\frac{1}{\sqrt{2}}$ gives the mean values as read by electro-dynamometers or thermal voltmeters. For the sake of completeness we will give the general analytical method of obtaining the equations in previous pages employed by Dr. J. Hopkinson.*

Taking the same symbols as before, Dr. Hopkinson writes the fundamental equation—

$$4\pi (N_1 i_1 + N_2 i_2) = \frac{lb}{a\mu} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where i_1 and i_2 are the currents in the primary and secondary circuits at any instant and b the corresponding total induction or number of lines of force traversing the whole section of the core, and l and a are the mean length and section of the magnetic circuit.

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* *Proceedings of Royal Society*, February, 1887, and *The Electrician*, Vol. XVIII., March, 1887, p. 420.

The two equations connecting the electromotive forces and currents are

$$e = R_1\, i_1 - N_1 \frac{d\,b}{d\,t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$0 = (r_2 + R_2)\, i_2 - N_2 \frac{d\,b}{d\,t} \quad . \quad . \quad . \quad . \quad (3)$$

r_2 being the resistance of the secondary circuit external to the transformer and R_2 that part interior. We suppose the part of the circuit which has a resistance r_2 to have no sensible inductance.

Eliminating b from (2) and (3) by multiplying (2) by N_2 and (3) by N_1 and subtracting the results, we have

$$N_2 e = N_2 R_1\, i_1 - N_1 (r_2 + R_2)\, i_2 \quad . \quad . \quad . \quad . \quad (4)$$

Combining this with the equation (1), viz.,

$$\frac{l\,b}{4\,\pi\,a\,\mu} = N_1\, i_1 + N_2\, i_2 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we have two simultaneous equations for i_1 and i_2, whence we obtain by solution

$$i_2\,[N_2^2 R_1 + N_1^2 (r_2 + R_2)] = \frac{l\,N_2 R_1\,b}{4\,\pi\,a\,\mu} - N_1 N_2\, e \quad . \quad (6)$$

and

$$i_1\,[N_2^2 R_1 + N_1^2 (r_2 + R_2)] = \frac{l\,N_1 (r_2 + R_2) b}{4\,\pi\,a\,\mu} + N_2^2\, e \quad (7)$$

by the substitution of the value of i_2 in equation (3) we obtain finally

$$\frac{db}{dt} = \frac{r_2 + R_2}{N_2} \left\{ \frac{\dfrac{l\,N_2 R_1\,b}{4\,\pi\,a\,\mu} - N_1 N_2\, e}{N_2^2 R_1 + N_1^2 (r_2 + R_2)} \right\} \quad . \quad . \quad (8)$$

Now, in complete-iron-circuit transformers the quantity $\dfrac{l\,b}{4\,\pi\,a\,\mu}$ is very small compared with the quantity $N_1 (r_2 + R_2)\, e$, for the permeability μ is very great, even when the induction b is small; hence, neglecting the first term in the bracket in the numerator in (8), we have

$$\frac{db}{dt} = -\frac{(r_2 + R_2)\, N_1\, e}{N_2^2 R_1 + N_1^2 (r_2 + R_2)} \quad . \quad . \quad . \quad . \quad (9)$$

but the secondary impressed electromotive force denoted by E_2 is equal to $N_2 \frac{db}{dt}$, and also to $(r_2 + R_2)\, i_2$;
therefore we have

$$i_2 = - \frac{\frac{N_2}{N_1} e}{\left(\frac{N_2}{N_1}\right)^2 R_1 + (r_2 + R_2)} \quad . \quad . \quad . \quad (10)$$

If e varies periodically, and the variation is not sufficient to carry b the induction up to the saturation point, then we may write for e and i_2 the maximum or the average values of the primary impressed E.M.F. E and the secondary current I_2, and we have

$$I_2 = \frac{E}{R_1 \frac{N_2}{N_1} + (r_2 + R_2) \frac{N_1}{N_2}} \quad . \quad . \quad . \quad (11)$$

and (11) is the same formula as (13), on p. 278, arrived at geometrically.

In connection with the foregoing equations the further remarks of Dr. Hopkinson are of importance.

We see, firstly, that the transformer transforms the primary potential E in the ratio of N_2 to N_1, and adds to the external resistance r_2 of the secondary a resistance equal to $\left(\frac{N_2}{N_1}\right)^2 R_1 + R_2$. This at once gives us the variation of potential caused by varying the number of lamps used. The phase of the secondary current is nearly opposite to that of the primary. In designing a transformer it is particularly necessary to take note of the assumption that the line integral of magnetic force is so small that it may be neglected. Taking e to vary harmonically, so that $e = E \sin p\, t$, we have, from (9), on p. 281,

$$\frac{db}{dt} = - \frac{(r_2 + R_2) N_1 E \sin p\, t}{N_2^2 R_1 + N_1^2 (r_2 + R_2)};$$

hence, by integration,

$$b = \frac{(r_2 + R_2)\, N_1\, E \cos p\, t}{N_2^2 R_1 + N_1^2 (r_1 + R_2)\, p} \quad . \quad . \quad . \quad (12)$$

If r_2 is infinite—that is, if the secondary circuit is open—then the maximum value of b, viz., B, is $B = \frac{E}{N_1 p}$; and, since $p = 2\pi n$, n being the frequency, we see that this maximum value of B, which must not exceed a certain limit, viz., a little below saturation, varies directly as E and inversely as n, and hence for a given primary pressure the higher the frequency the less in general, neglecting any question of hysteresis, can be the induction.

§ 14. **Effects of Saturation and Magnetic Hysteresis.**—The above approximations, though ample for practical work, give

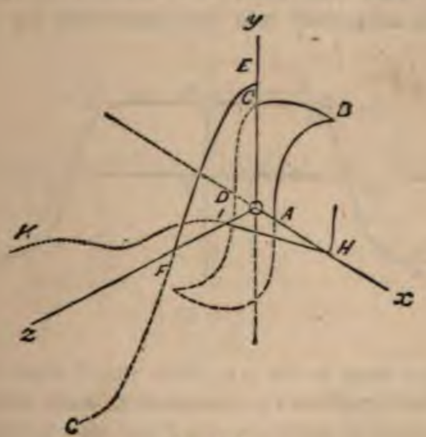

Fig. 49.

no account of what happens when transformers are worked so that the iron core is saturated, or how energy is wasted in the iron core by the continual reversal of its magnetism. The amount of such waste is easily estimated from Ewing's results, when the maximum value of the magnetic induction B is known; but it is instructive to proceed to a second approximation, and see how the magnetic properties of the iron affect the value and phase of i_1 and i_2. We can, in a second approximation, substitute in the preceding equations the value for the magnetic force deduced from the value of b, or the induction furnished by the approximate equation (10). In the accompanying diagram (Fig. 49) O x represents the magnetising force, O y the magnetic induction, and O z the time. The curve A B C D, drawn on the plane of $x\,y$, represents the relation of magnetic force and magnetic induction for a complete magnetic cycle, the curve E F G the induction as a function of the time, and H I K the deduced relation of magnetic force and time. We may substitute the values of the force obtained from this curve for the value of $\frac{b}{a\,\mu}$ in the equations (6) and (7), p. 281, and so obtain the values of i_2 and i_1 to a higher degree of approximation. If the values of the force were expressed by Fourier's theorem in terms of the time, we should find that the action of the iron core introduced into the expressions for i_2 and i_1, in

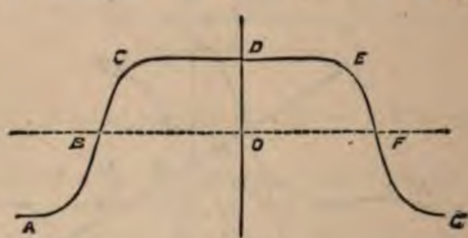

FIG. 50.

addition to a term in cos $p\,t$, which would occur if magnetic force and induction were proportional to each other, terms in sines and cosines of multiples of $p\,t$, and also a term in sin $p\,t$. It is through this last term that the loss of energy by hysteresis comes in.

A particular case in which to stay at a first approximation would be very misleading is worthy of note. Let an attempt be made to ascertain the highest possible values of the induction by using upon a transformer a very large primary current, and measuring the consequent mean square of potential in the secondary circuit by means of an electrometer or a thermal voltmeter. The value of b will be related to time in some such manner as indicated by the curve in Fig. 50. For simplicity, assume the time variation of induction to be as indicated by the firm line curve in Fig. 51. The resulting relation of impressed electromotive force in the secondary, and, therefore, practically of secondary current and time, will be indicated by the dotted curve. The mean square observed will be proportional to the product of M L and $\sqrt{LP}$, but M L $\sqrt{LP}$ is proportional to E L. Hence, the potential observed will vary inversely as $\sqrt{LP}$ even though the maximum induction remain constant. If, then, the maximum induction be deduced on the assumption that the induction is a simple harmonic function of the time, results may be obtained vastly in excess of the truth.

Following Mr. Kapp's geometrical diagram, we can proceed to show how the diagram in Fig. 47 becomes modified if the saturation of the core is allowed to take place—that is, if the

FIG. 51.

exciting or resultant magnetising force is carried up during the phase to such a point that the core becomes saturated. The effect of magnetic saturation of the core is to render the induced electromotive forces in the primary and secondary circuits not strictly proportional to the effective exciting power but rather smaller, and will be felt during that portion of the phase of the magnetisation when the core is approaching saturation, but not just when the magnetisation is reversing sign or direction.

Let O K, Fig. 52, represent a magnetisation curve for the core, so drawn that whilst horizontal distances, measured outwards and leftwards from O along O X, represent magnetising ampere turns, vertical ordinates downwards represent corresponding values of $p\ N_2 B$, where B is the total induction through the core. This can always be done if the curve is known in terms of magnetising force and magnetic induction. We, have, then, that corresponding to a resultant or effective ampere turns O B, the value of the maximum secondary impressed electromotive force is B K = O V[1], and this smaller than would be the case if the induction at or about an exciting power, O B, were proportional to the induction for proportionately less exciting power We can then obtain the inductive electromotive force in the

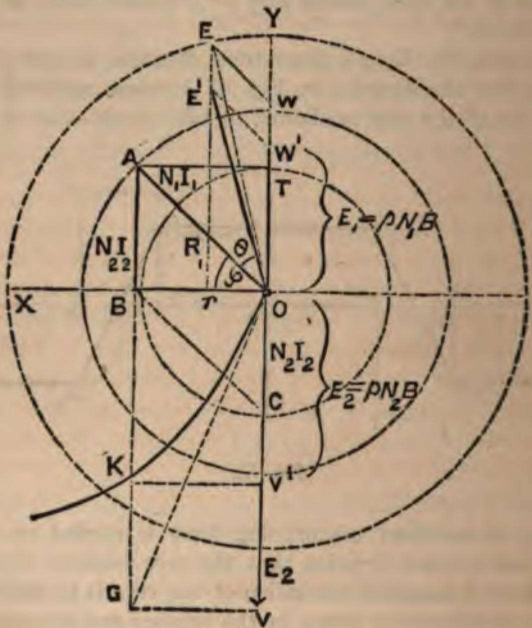

Fig. 52.

primary by changing the magnitude of O V[1] in the ratio of N_1 to N_2, and we get a value represented by O W[1] which is also smaller than O W, corresponding to very incomplete saturation, as in the former case. Hence, concluding the construction as before for O E, or the impressed electromotive force, we find a line, O E[1], which is not only smaller than O E, but inclined at a less angle to O A. It follows that the effect of the diminished induction is to reduce the impressed electromotive force corresponding to a definite primary current, to decrease the lag of the primary current behind the impressed E.M.F., and to reduce the secondary current and E.M.F. As, however, the effect of saturation is not felt except near the maximum points of the phase of the magnetisation, the result will be to cause a periodic change in the angles of lag, and also in the primary current or primary impressed electromotive force depending on which variable is kept constant.

We have, then, to consider the general effects of hysteresis. Physically speaking, the effect of this lag of magnetic induction behind magnetising force when the iron is carried round a magnetic cycle is to make a dissipation of energy depending partly on the maximum value of the induction and partly on the rate at which the cycle is performed. This distinction of static hysteresis and viscous hysteresis might be illustrated by the mechanical analogue of lifting a weight in a viscous fluid. Apart from fluid resistance the work done in lifting the weight against gravity, say one hundred times, is a hundred times the work required to be spent to lift it once; but if fluid resistance comes into play, and if this varies as the square of the velocity of the moving body, then the total work done in lifting the weight through the fluid will be dependent also upon the rate at which the cycle is performed. Reckoning from certain experimental data obtained by Prof. Ewing, Mr. Kapp has calculated the hysteresial dissipation of energy in transformer cores when magnetisation is reversed in them 200 times per second—that is, when 100 complete magnetic cycles are performed per second, and when the maximum induction reaches to a figure given in the first column. The results are stated below in watts and horse-power per ton of iron.

Maximum Induction in the Iron Core.	Watts per Ton of Iron.	H.-P. Wasted in Heat per Ton of Iron.
2,000	650	0·87
3,000	1,100	1·48
4,000	1,650	2·21
5,000	2,250	3·02
6,000	2,900	3·89
7,000	3,750	5·03
8,000	4,450	5·97
9,000	5,550	7·43
10,000	6,650	8·90

In these figures, however, the "viscous" part of the hysteresis is not taken into account for want of sufficient data, but it is probable that a notable increase would have to be made on this account. These figures, however, are sufficient to show that, quite apart from other considerations, it is not advisable to press the induction in transformer cores beyond about 6,000 or 7,000 C.-G.-S. units. The waste of energy in internal heating is a thing quite apart from eddy electric currents generated by induction, and cannot be subdued by lamination of the core. The effect of hysteresis in producing a lag of induction behind magnetising force has been indicated graphically in Fig. 23, p. 65, but for the pre-

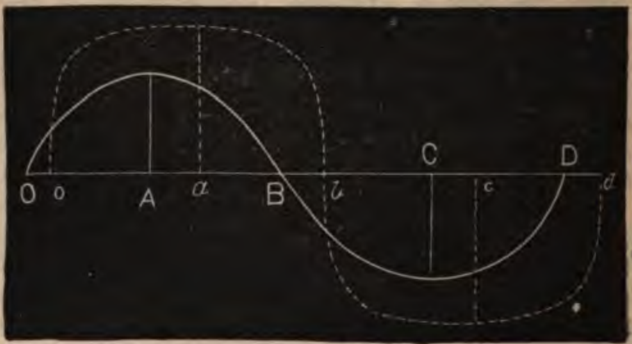

FIG. 53.

sent purpose it will be more instructive to illustrate it in another manner. Taking the values of magnetising force and corresponding magnetic induction given by Prof. Ewing (Fig. 2, Plate 57, *Phil. Trans. Royal Soc.*, 1885), let the values of the magnetising force be taken off as ordinates on a simple sine curve (*see* Fig. 53), and let corresponding values of the induction be set off to any scale on the same ordinates, and the time change of induction represented by the dotted line. The variations of the ordinate of the firm line sine curve may represent, then, a magnetising force varying in a simple periodic manner with the time, and the dotted curve will represent the corresponding variations of the magnetic induction for the particular ring experimented with; and supposing the changes of force slow, and all effect of eddy current annulled, we see that the induction curve is a kind of square-shouldered curve which is shifted backwards in phase relatively to the magnetic force curve.

In this case it will be observed that the induction has been carried well up on to the "knee" of the magnetising curve, and hence the bluntness, truncated, or square-shouldered form of the time-induction curve. Consider now for one moment how the induction lag affects the phase of the secondary current if we suppose this variation of induction to be operative in so producing a current in a secondary circuit wound on this core. Obviously, the instant of maximum rate of change of induction which corresponds to the points *b* and *d*, where the induction curve crosses the time line and has its greatest *slope*, is behind the instant of maximum of rate of change of magnetic force, which corresponds to the points B and D, where the force curve has its greatest slope and crosses the time line. The secondary impressed electromotive force depends on and coincides in phase with the rate of change of induction, and therefore is determined by the slope of the induction curve. Hence, if we draw the curve which represents the time rate of change of the ordinate of the induction curve, or the secondary electromotive force, we find it to be a curve consisting of tall peaks separated by intervening flat valleys, and the time of the maximum of this secondary impressed electromotive force has a lag introduced into it by reason of the hysteresis. On the assumption that the secondary-circuit inductance is negligible we see that the secondary current is no longer in quadrature with, or 90deg. behind, the magnetising force, as before assumed, but lags behind it by an angle *slightly greater*, on account of the lag of the induction behind the magnetising force due to hysteresis.

This is the place to remark that some physicists doubt the existence of a true magnetic time lag, and consider that whatever effects have been observed which experimentally point to such a lag of magnetisation behind magnetising force are the result of hysteresis and of eddy currents, and that the application of a magnetising force whilst it is, say, increasing, produces in the iron reversely directed surface eddy currents introducing a reverse or opposed magnetising force on the more deeply seated layers of iron, and that until these eddy currents subside the impressed magnetising force is unable to produce its full effect in magnetisation. On the other hand, it seems clear that hysteresis is an effect entirely apart from any question of surface eddy currents, and that whatever may be its nature it does render the upward and downward magnetic induction curve separate. Hence, to this degree it shifts the curve of induction backwards relatively to the magnetic force curve and introduces an added lag into the phase of the secondary current.

Mr. Blakesley (*Phil. Mag.*, July, 1888) has shown how this induction lag can be measured by the use of three dynamometers.

We have shown on p. 127, § 16, Chap. III., that if a simple periodic current whose maximum value is I_1 is passed through an electro-dynamometer, the reading of the instrument is proportional to $\frac{I_1}{\sqrt{2}}$, that is to say, it gives a reading equal to that which would be given by a steady current of this value passed through it.

Again, on p. 119 we have shown that the mean value of the product of two simple periodic functions differing in phase is equal to their product and that of half the cosine of the angle of lag.

Accordingly, if two simple periodic currents are passed through an electro-dynamometer, one current going through the fixed coil only and one through the movable coil only, the reading of the instrument is proportional to half the product of the current strengths and to the cosine of the angle of phase difference.

Let three dynamometers be so joined up to a transformer that the first is traversed by the primary current alone, the second by the secondary current alone, and the third traversed by the primary current going through its fixed coil and the secondary current through its movable coil; then the torsion reading of the first dynamometer being, say, $\alpha°$, and the second $\beta°$, and the third $\gamma°$, we have that α is proportional to $\frac{I_1^2}{2}$, β to

$\frac{I_2^2}{2}$, and γ to $\frac{I_1 I_2}{2}\cos\theta$, where θ is the angle of phase difference of the currents. Hence, we may write

$$A\,\alpha = \frac{I_1^2}{2}$$

$$B\,\beta = \frac{I_2^2}{2}$$

$$C\,\gamma = \frac{I_1 I_2}{2}\cos\theta.$$

Hence,
$$\cos\theta = \frac{C\,\gamma}{\sqrt{A\alpha - B\beta}},$$

A, B, and C being the instrumental constants. This method, which is due to Mr. Blakesley, enables us to measure the angle

FIG. 54.

of lag of one current behind another. Let, then, the magnitude of the primary and secondary ampere turns in the transformer be known, and the angle of lag, as above, be measured. Draw a line A B (Fig. 54), to represent the primary ampere turns (maximum value), and a line, A C, to represent the secondary ampere turns. Set A B and A C at an angle equal to the measured lag angle of the currents. From B drop a perpendicular on A C produced. Join C B.

Then
$$A\,D = A\,B\cos\theta,$$
$$= N_1\,I_1\cos\theta,$$

and
$$A\,C = N_2\,I_2.$$

Hence, if $N_2 I_2$ is less than $N_1 I_1 \cos\theta$ it indicates that B C is not in quadrature with A C—that is, the resultant magnetising force is not in quadrature with the secondary current, and the angle D B C is the angle of magnetic lag.

The condition of the existence of magnetic lag is therefore that $CA < AB \cos\theta$, or in terms of the dynamometer observations,

$$N_2\sqrt{2B\beta} < N_1\sqrt{2A\alpha}\frac{C\gamma}{\sqrt{A\alpha \cdot B\beta}},$$

or

$$B\beta < \frac{N_1}{N_2}C\gamma.$$

Hence the lag, if it exists, can be detected by two dynamometer observations. We can measure it thus:—Call the angle D B C ϕ. Then

$$\tan\phi = \frac{CD}{BD} = \frac{DA - AC}{AB\sin\theta},$$

$$= \frac{\frac{DA}{AB} - \frac{AC}{AB}}{\sqrt{1-\cos^2\theta}},$$

$$= \frac{\cos\theta - \frac{N_2 I_2}{N_1 I_1}}{\sqrt{1-\cos^2\theta}},$$

$$= \frac{\frac{C\gamma}{\sqrt{A\alpha B\beta}} - \frac{N_2}{N_1}\sqrt{\frac{B\beta}{A\alpha}}}{\sqrt{1-\frac{C^2\gamma^2}{A\alpha A\beta}}},$$

$$= \frac{C\gamma - \frac{N_2}{N_1}B\beta}{\sqrt{A\alpha B\beta - C^2\gamma^2}}.$$

Mr. Blakesley made observations as above on a Kapp transformer, in which $N_1 = 100$, $N_2 = 12$, and the result of seven observations showed that ϕ was an angle not far from 5° 30′ (*Phil. Mag.*, July, 1888). It seems evident, then, that whatever causes an absorption of power in changing magnetisation causes magnetic lag—in other words, causes the time of maximum rate of change of magnetic induction to lag behind the time of maximum rate of change of magnetising force, and hence makes the phase of the maximum secondary impressed electromotive force to be something more than 90deg. behind the maximum of the resultant magnetising force, on the assumption that this latter varies according to a simple periodic law. Having in any case found the value of the magnetic lag, it is possible to correct the diagram in Fig. 52, and to make the angle B O V′ greater than a right angle by the proper amount. It is, then, manifest that this will to an equal degree diminish the angle A O W, and make O E greater in magnitude and bring O E slightly more into consonance with O A. In other words, the effect of magnetic lag is shown by bringing the impressed electromotive force more into consonance with the primary current, and increasing, therefore, as it should do, the work spent on the primary coil. This additional demand for work represents the dissipation of energy going on in the core by reason of the hysteresis, or whatever causes the lag.

It is necessary to distinguish between the lag of induction behind magnetising force which is thus measured and which is the evidence of hysteresis, and anything like a true delay in the appearance of magnetisation in iron on the imposition of a definite magnetising force. Opinion on the whole tends in the direction that a true magnetic lag has not yet been observed other than that which is due to the gradual evanescence of the reverse magnetising force of the eddy currents produced in the skin of the iron, and which, as long as they last, oppose the magnetisation of the deeper layers of the iron, and cause a delay in the appearance of the full magnetisation. Experiment shows that these eddy currents exist even in cores of divided iron. When the alterations of the exciting current became very rapid, these eddy currents entirely screen the interior of the iron and render it practically non-magnetic. Further allusion will be made to this point under the head of *electrical oscillations.*

§ 15. **Characteristic Curves of the Series and Parallel Transformer.**—The characteristic curve of a transformer is a curve so drawn that its ordinate and abscissa at any point represent the secondary electromotive force and secondary current when the resistance of the secondary circuit has a certain definite value. Consider first the case of a transformer used in series with others; that is, having the primaries of all traversed by the same primary current, which is maintained constant. We have then to see how the secondary currents and secondary electromotive forces are related to one another when the secondary resistance of each is changed in any way. Referring back to Fig. 47, on p. 274, we see that A B represents the secondary ampere turns, and that therefore a certain fraction of A B may be taken to represent the secondary current. Also O B represents the resultant magnetising force, and may be taken therefore, when we are well within the limits of saturation, as proportional to the induction B. But the maximum secondary electromotive force is also proportional to B, for it is equal to $p N_2 B$. Hence we may say that a certain definite fraction of O B represents the secondary impressed electromotive force (maximum value). Accordingly, if the point A is taken at various positions round the circle of radius O A, the corresponding variations of A B and O B will indicate the manner in which secondary current and secondary electromotive force vary, and if we plot a curve such that its abscissa x is always a definite fraction, say $\frac{1}{p}$ of A B, and the corresponding ordinate y is always a definite fraction, say $\frac{1}{q}$ of A B, and since $x=\frac{1}{p}$ A B, and $y=\frac{1}{q}$ O B, and since $(A\,B)^2+(O\,B)^2=(A\,B)^2=$ *a constant quantity*, we have $p^2\ x^2+q^2\ y^2=a$ *constant* as the equation to the characteristic curve, and this is an ellipse.

If, then, O a (*see* Fig. 55) is taken on a line, O X, to represent the secondary current, and if O a is proportional to the secondary current, and $a\,b$ taken perpendicular to O a represents the corresponding secondary electromotive force, the various positions of the extremity b, as O a has given to it its various values, will mark out a curve which is an ellipse, and which is the characteristic of a series transformer.

The tangent of the angle a O b represents of course the total resistance of the secondary circuit. On this subject Mr. Kapp remarks (*The Electrician*, Vol. XVIII., p. 570, 1887) :—

"The characteristic of a series transformer thus obtained is very instructive. It shows at a glance what alterations take place in the E.M.F. and the current (this of course refers to their maxima, but the mean values are proportional thereto) in the secondary circuit if its resistance be varied, say, by switching lamps in or out. For very large values of the resistance the alteration of current is proportionately much greater than that of E.M.F., whilst for small resistances the alteration of current is small compared with that of E.M.F. Thus, if only a few glow lamps are being fed in parallel from the secondary circuit, the switching out of these lamps will not materially

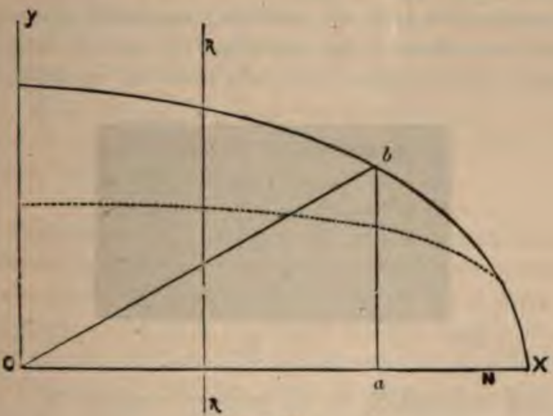

FIG. 55.

increase the pressure in the circuit; but if a sufficient number of lamps be switched on so as to fully utilise the energy of the transformer, then the pressure will fall considerably, and the lamps will become dim. On the other hand, we might burn a limited number of glow lamps in series, and by short-circuiting, one after the other the current would not be materially increased, but also in this case the energy converted would be much less than could be obtained with the given transformer if worked in a condition represented by the middle portion of the characteristic. It is interesting to note that even if the secondary coil be completely short-circuited the current in it does not become very great. Roughly speaking, it is twice the current for which the transformer develops its maximum energy. For commercial reasons it is necessary to work transformers at or near the maximum output, and a glance at the curve will show that this class of apparatus (series transformer), with a core sufficiently large to avoid approach to saturation, can never be self-regulating if so used. It can only be made self-regulating with a non-saturated core when worked near the extremities of the characteristic with very small secondary current or very low secondary electromotive force, both of which conditions are uncommercial."

If the condition of supply is constant electromotive force on the primary circuit, then the characteristic may be found from the consideration that the secondary impressed electromotive force has been shown to be approximately equal to the primary impressed electromotive force multiplied by the ratio of the

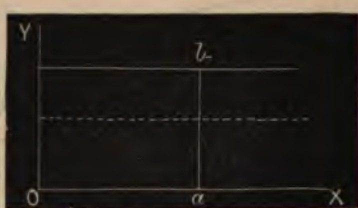

FIG. 56.

secondary windings N_2 to the primary windings N_1. Hence, E_2 is a constant quantity and independent of the secondary current. Accordingly the characteristic is nearly a straight line (Fig. 56) drawn parallel to O X, the axis of current, and within limits a variation of secondary current does not affect the secondary electromotive force. A parallel transformer will then be perfectly self-regulating under constant primary electromotive force.

With respect to the effect of saturation on the characteristic, it is easy to see that in the case of the parallel transformer the saturation of the core causes the induction to be less relatively to the resultant magnetising force than it would be if saturation had not intervened, and hence the secondary electromotive force, whilst being smaller, will still be independent of the magnitude of the secondary current; in other words, the characteristic will still be a straight line parallel to the axis of current, but will have to be drawn nearer the axis of the current. In the case of the series transformer, saturation will have the same effect, and will cause the characteristic to be still an elliptical curve, but having a lesser minor semi-axis, and it will be represented as by the dotted curve in Fig. 55. In the case, therefore, of the parallel transformer, to work with a high degree of induction is still compatible with the maintenance of nearly constant secondary electromotive force under varying loads. If, however, the supply is on the series system, with constant primary current, we see that from the flatter elliptical curve the variation of secondary electromotive force is smaller under small currents or loads. A series transformer, with a small and over-saturated core, might therefore be made to regulate within somewhat wider limits than one where the core is large, but the regulation can never be perfect, and the disadvantages of over-saturation in respect of hysterial energy waste would over-balance any such slight advantage.

§ **16. Efficiency of Transformers.**—The electrical efficiency of a transformer is defined to be the fraction or percentage of the whole energy, supplied in a given time to the primary circuit, which appears in the form of electric current energy in the secondary circuit. Part of the energy supplied is dissipated by eddy currents in the core, some by magnetic friction or hysteresis, and some in heating the primary circuit. Of that portion which appears in the secondary circuit part is expended in heating the portion of the secondary circuit which consists of the actual coils of the transformer, and part is utilised in the external portion of the secondary circuit, which will generally consist of lamps or transforming devices.

We proceed first on the assumption that the iron core is a complete circuit, and that the induction is small enough to render hysteresis negligible, and that the external secondary circuit consists of lamps, and therefore practically has very small self-induction. Prof. Forbes has given a solution of the transformer problem on the same lines as that of Dr. Hopkinson.*

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* See *Journal of Society of Telegraph-Engineers and Electricians*, Vol. XVII., Part LXXI., p. 153.

In our notation his results are as follows:—If w_1 be the work done in unit time in the primary circuit, and w_2 that done in the secondary, and hence if the efficiency be ϵ where $\epsilon = \frac{w_2}{w_1}$, he deduces the relation that

$$\epsilon = \frac{1}{1 + \frac{N_2^2 R_1}{N_1^2 R_2} + \frac{R_1 R_2}{N_1^2 N_2^2} \frac{\rho^2}{64\pi^2 n^2}},$$

where ρ is the "magnetic resistance" of the iron core, or,

$$\rho = \frac{\text{mean length of lines of induction in the iron}}{\text{cross sectional area of iron} \times \text{permeability}},$$

N_1 and N_2 being turns of wire, R_1 and R_2 resistances of primary and secondary, and n the number of complete oscillations of the current.

If $\rho = 0$, that is if the permeability is very large, we have the same expression for the efficiency as given in (15) on page 280. If the values of N_1, N_2, R_1, R_2, pertaining to a modern transformer, are inserted, it is found that $\frac{N_2^2 R_1}{N_1^2 R_2}$ is a quantity generally less than ·01.

Hence, the efficiency for a transformer of given turns will depend on ρ and on R_2—that is, on the secondary resistance as varied by the lamp load.

Suppose the transformer to carry a lamp-load on its secondary of lamps arranged in parallel, then the extinction of lamps will *increase* R_2, and by increasing, therefore, the numerator of the third term in the denominator of the fraction expressing the efficiency it will *diminish the efficiency*. The efficiency is therefore *less* for light load than for heavy load of parallel lamps. Again the effect of varying speed of alternation is seen, for under given circumstances of load and windings, the increase of n increases the efficiency by making the third term smaller, but n may be diminished without loss of efficiency if ρ or the magnetic resistance is increased; hence, for working economically with small frequency of oscillation large massive cores are essential, but by increasing the frequency the core mass may be diminished without loss of efficiency.

Prof. Forbes also showed (*loc. cit.*) that when the secondary circuit is unloaded—that is, when R_2 is infinite—the waste of energy in the transformer is given by the expression

$$\text{waste} = \frac{B^2}{2} R_1 \left(\frac{\rho}{4\pi N_1}\right)^2,$$

and is, therefore, proportional to the square of the magnetic resistance of the core, B being the total maximum induction through the core.

Practical measurements on the efficiency of a transformer have been given by Prof. Ayrton, made by a method first apparently suggested by Dr. Louis Duncan, of John Hopkins University, Baltimore (see *Electrical Review*, July 29, 1887, Vol. XXI., p. 116), in which the waste of energy in the transformer was obtained from calorimetric measurements.*

The efficiency was measured for various loads on the secondary circuit, and the efficiencies expressed as percentages plotted as ordinates to abscissæ representing the secondary output in watts. The results were as follows:—

Efficiency of Kapp and Snell Transformer.

Primary Circuit.		Secondary Circuit.		Efficiencies in percentages.
Terminal volts.	Current in amperes.	Terminal volts.	Current in amperes.	
184·3	0·8	94·5	0	0
184·6	2·3	96·8	3·3	80·1
187·6	3·32	97·7	5·09	85·8
186·1	8·31	94·3	14·87	96·2
185·6	16·14	91·44	30·59	95·0

The efficiency-curve from these tabular values is given in Fig. 57, and shows that the efficiency has a *maximum for a certain value of the secondary resistance* R_2. This arises from the fact that the continual increase of lamp-load by diminishing R_2 at first affects chiefly the third term in the denominator of the

* These practical measurements will be discussed in detail in Vol. II. of this work. For details of Prof. Ayrton's results, see *Journal of Society of Telegraph-Engineers*, Vol. XVII., 1888, p. 168. These experimental results were obtained from a small Kapp and Snell transformer by students working at the Central Technical Institution, London.

fraction expressing ϵ on p. 298, but that as R_2 decreases, the decrease in the value of the third term is overbalanced by the increase of the second term, and the efficiency then begins to fall again.

Captain Cardew has also made experimental measurements of the efficiencies of a Gaulard and Gibbs transformer at various loads. In this case the efficiency was calculated from the formula—

$$\epsilon = \frac{\text{secondary current} \times \text{secondary E.M.F.}}{\text{primary current} \times \text{primary E.M.F.} \times \cos\theta},$$

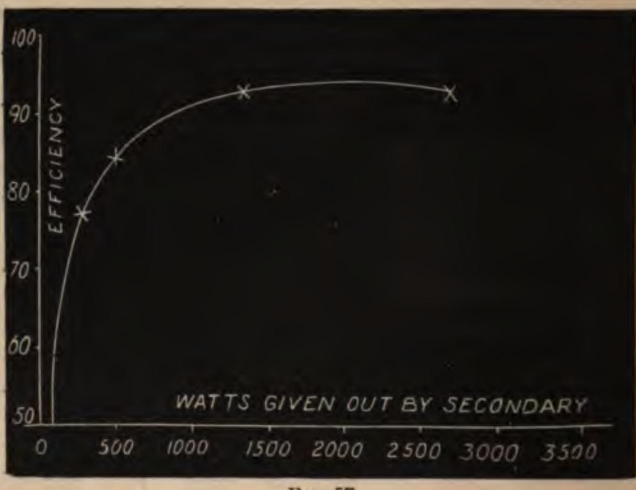

Fig. 57.

where θ is the angle of lag of the primary current behind the primary impressed electromotive force and θ was calculated by the formulæ (10) on page 276.*

The transformer was a Gaulard and Gibbs transformer. The primary and secondary windings were in the ratio of 4 : 5, and the primary resistance was ·023 ohm and secondary ·036.

The external secondary circuit consisted of incandescent lamps. The terminal volts of primary and secondary were taken by a Cardew voltmeter, and the currents, primary and secondary,

* See *Electrician*, July 8, 1887, Vol. XIX., p. 185.

by Siemens' dynamometer. The angles of lag of primary current behind impressed electromotive force (θ) and of magnetisation behind primary current (ϕ) were calculated from the formulæ as on pp. 276 and 277, and the results tabulated as follows :—

Tests of Gaulard and Gibbs Transformers.

Lamp-load on secondary.	Ampere Currents.		Terminal Volts.	
	Primary Circuit.	Secondary Circuit.	Primary Circuit.	Secondary Circuit.
23	56·7	41·83	43·5	51·5
15	49	34·04	52	63·5
12	46·4	29·81	56	68·5 (?)
9	42·9	20·39	50	60·6
6	41·1	16·68	56	69·5
3	39·6	7·9	64	79
0	41·8	·2	61·2	77

Angles of Lag and Efficiencies.

Lamp load on secondary.	θ	ϕ	$\theta+\phi$	ϵ in per cent.
23	22° 8′ 0″	67° 12′ 0″	89° 20′ 0″	94·3
15	29 7 0	60 16 0	89 23 0	97·15
12	35 58 45	53 25 25	89 24 10	99·39 (?)
9	52 39 50	36 26 40	89 5 30	95
6	58 41 0	30 29 0	89 10 0	96·94
3	74 46 10	14 26 25	89 12 35	93·74
0	88 45 25	0 20 35	89 6 0	27·76

The efficiencies (ϵ) are only approximations. An estimated amount of about $2\frac{1}{2}$ per cent.* has been deducted from the calculated efficiencies to allow for waste by magnetic friction and power wasted in the core, and the values of ϵ are the approximate external efficiencies resulting. The values of the angle of lag $\theta+\phi$ show that the magnetisation is very nearly 90 deg. in phase behind the impressed electromotive force, and

* This estimate of the dissipated energy is certainly rather too small. The researches of Prof. Ferraris indicate a higher value, and probably a 5 or 7 per cent. subtraction would not be inappropriate in this case. Hence the tabulated efficiencies are all too great.

that hence this last must be in opposition as regards phase to the secondary current. The values of ϵ also show a falling off in efficiency of a considerable amount when the load was made very light, for then only the secondary current taken off was by the Cardew voltmeter employed to measure the terminal secondary volts.

§ **17. Open Iron Circuit Transformers.**—In some cases transformers have been constructed with iron cores not forming a complete iron circuit. In this case the magnetisation curve of the iron will, on account of the resistance of the air part of the magnetic circuit, be a curve rising up to a "knee" at a rather gentle slope, and for a considerable period below saturation will practically be a line not far from straight. In this case we might consider the coefficients of self and mutual induction of the circuits to have large but tolerably approximately constant values during the range of the magnetising force through which it is advisable to work.

In this case the theory of the transformer can be treated on the assumption that the values of these constant but large induction coefficients can be known. We can follow, then, a method due to Maxwell, in which equations are obtained for the current and secondary electromotive force in terms of these supposed known coefficients. As a first approximation suppose, as before, the primary impressed electromotive force at any instant t is e_1, and if

$$e_1 = E_1 \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

E being as usual the maximum of the impressed E.M.F. and $p = 2\pi n$, n being the frequency, we have then as the equations for the currents

$$L\frac{dx}{dt} + M\frac{dy}{dt} + Rx = e_1 \quad . \quad . \quad . \quad . \quad (2)$$

and

$$N\frac{dy}{dt} + M\frac{dx}{dt} + Sy = 0 \quad . \quad . \quad . \quad . \quad (3)$$

precisely as on p. 182, L, R, and x being the primary inductance, resistance, and current at any instant, N, S, and y being the corresponding secondary circuit values, and M the coefficient of mutual inductance of the circuits, the equations (2) and (3) expressing the fact that the impressed E.M.F. at any instant balances the effective and inductive electromotive forces. In order to separate out the expressions for x and y we proceed as on pp. 182 *et seq.*

Differentiate (2) with respect to time, and we get

$$L\frac{d^2x}{dt^2} + M\frac{d^2y}{dt^2} + R\frac{dx}{dt} = \frac{de_1}{dt} \quad . \quad . \quad . \quad (4)$$

Since e_1 is a simple periodic function, it follows that the currents will, after a small interval, fall into step with the impressed electromotive force, and vary in a simple periodic manner. Hence, since x may be taken to be a simple sine function, $\frac{d^2x}{dt^2}$ is one also, and if we write

$$x = X \sin(pt + \theta),$$

it follows that

$$\frac{d^2x}{dt^2} = -p^2x.$$

On this ground we can write (4),

$$\frac{de_1}{dt} + Mp^2y + Lp^2x - R\frac{dx}{dt} = 0 \quad . \quad . \quad . \quad (5)$$

Now multiply (5) all through by R, multiply (2) all through by Lp^2, and add the resulting equations, we arrive at

$$\left(R^2 + p^2L^2\right)\frac{dx}{dt} = R\frac{de_1}{dt} + Lp^2e_1 + Mp^2\left(Ry - L\frac{dy}{dt}\right) \quad (6)$$

Multiply every term of (6) by $\frac{Mp}{\sqrt{R^2 + p^2L^2}}$, and write as an abbreviation for this last factor the symbol k, and let $k^2R = r$, $k^2L = l$. Write also for the expression

$$-\frac{k^2}{M}\left(\frac{R}{p^2}\frac{de_1}{dt} + Le_1\right)$$

the single symbol e_2. We shall find that with these abbreviations and reductions the equation (6) reduces to

$$M\frac{dx}{dt} = ry - l\frac{dy}{dt} - e_2 \quad . \quad . \quad . \quad . \quad (7)$$

Between (7) and (3) eliminate $M\frac{dx}{dt}$, and we have

$$(N - l)\frac{dy}{dt} + (S + r)y = e_2 \quad . \quad . \quad . \quad . \quad (8)$$

Since $e_1 = E_1 \sin p\,t$, it follows that

$$-\frac{k^2}{M}\left(\frac{R}{p^2}\frac{d\,e_1}{d\,t} + L\,e_1\right) = e_2$$

may be transformed into

$$-\frac{k^2}{M}E_1\left(\frac{R}{p}\cos p\,t + L\sin p\,t\right) = e_2,$$

and this by Lemma on p. 132 may be written

$$-k\,E_1\sin(p\,t+\phi) = e_2 = E_2\sin(p\,t+\phi),$$

which shows that E_2 is numerically equal to $k\,E_1$. Hence the equation (8) may be written finally

$$(N-l)\frac{d\,y}{d\,t} + (S+r)y = -k E_1\sin(p\,t+\phi).$$

This last equation shows us that the current which *does* flow in the secondary circuit under the inductive action on the primary current produced by a primary electromotive force, E_1, might be exactly imitated if we removed the primary circuit, diminished the secondary inductance by a quantity, l, equal to $k^2 L$, increased the secondary resistance by a quantity, r, equal to $k^2 R$, and impressed an electromotive force equal to $k\,E_1$ directly on the secondary circuit. If the frequency of the alternations, or the permeability of the magnetic circuit, is increased so that the primary ohmic resistance R vanishes in comparison with the value of $L\,p$, then k becomes equal to $\frac{M}{L}$, or, which is the same thing, to the ratio of the secondary to primary windings, viz., to $N_2 \div N_1$. In this case the E.M.F. is transformed in the ratio of the windings.

If, in addition, there are about equal weights of primary and secondary wire on the core, we have also the relation

$$\frac{R}{S} = \left(\frac{N_1}{N_2}\right)^2 = \frac{1}{k^2},$$

connecting the resistances and turns on the primary and secondary, and since the inductances of the coils are in the ratio of the square of the number of turns, it follows that

$$\frac{L}{N} = \left(\frac{N_1}{N_2}\right)^2 = \frac{1}{k^2}.$$

Under these conditions $L k^2 = N = l$ and $R k_2 = S = r$. Hence, the result is that the self-induction is annulled and the secondary resistance doubled, and the current in the secondary will be equivalent to the current flowing in a conductor of double its ohmic resistance under an electromotive force equal to

$$\frac{N_2}{N_1} E_1.$$

§ **18. Experimental Results of Prof. Ferraris.**—Prof. Galileo Ferraris has published an extensive and complete monograph on the experimental results obtained with open-circuit transformers of the type designed by Gaulard and Gibbs.*

The experiments here described were performed with a pattern of Gaulard and Gibbs transformer, consisting of a simple straight core. The primary and secondary circuits were formed of flat discs or rings of thin copper, and formed two interleaved circuits, the separate turns of the spirals being insulated by paraffined paper. The iron core was cylindrical. The external diameter of the spirals was 114 millimetres, or about $4\frac{1}{2}$ inches, and the internal 54 millimetres. The number of turns on each coil was 455. The primary and secondary circuits had resistances of 0·276 and 0·285 ohms respectively. The currents were measured by dynamometers of the Siemens pattern, and of which the coefficients of induction were negligible in comparison with those of the transformer circuits. The difference of phase of the primary and secondary current was determined by Blakesley's method, with three dynamometers (*see* Fig. 58). The first experiments consisted in determining the constants of the electro-dynamometers, so that the observed readings α, β, γ could be interpreted to give the ratios of the mean current or maximum currents during the phases. The observed readings of the dynamometers are proportional to the mean of the squares of the various values through which the current runs in its phase, and this is proportional to half the value of the square of the maximum value if the current curve is sinoidal. Hence, if the

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* See *La Lumière Électrique*, Vol. XVI., p. 399, 1885, and Vol. XXVII., p. 518, 1888, and *Electrical Review*, Vol. XVI., pp. 256, 343, 1885; also *Electrical Review*, Vol. XXII., pp. 111, 132, 156, 191, 220, 252; also *Sulle Differenze di Fase della Correnti e sulla Dissipazione di Energia nei Transformatori*, by Prof. Galileo Ferraris (Turin, 1887).

same current is sent through the three instruments in series, and if I_0 is its maximum value, and if $\alpha_0, \beta_0, \gamma_0$ are the simultaneous readings, then

$$\alpha_0 = h_1 I_0^2, \quad \beta_0 = h_2 I_0^2, \quad \gamma_0 = h_3 I_0^2,$$

h_1, h_2, h_3 being the respective instrumental constants. If, then, the primary and secondary currents are sent separately through the first and second dynamometers, and combined through the third to determine the phase difference ϕ of the current, and if

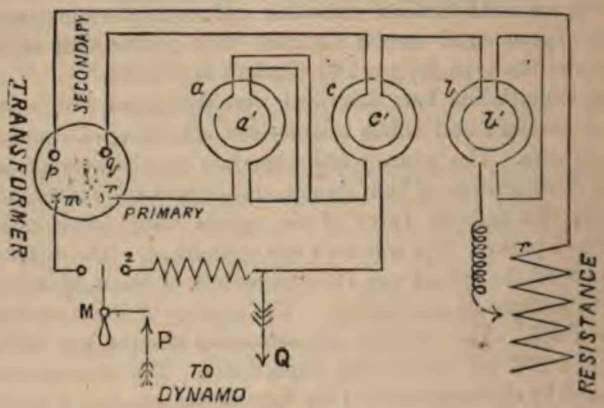


FIG. 58.

A and B are their maximum values, and if α, β, γ, are the instrumental readings, we have

$$\alpha = h_1 A^2, \quad \beta = h_2 B^2, \quad \gamma = h_3 A B \cos\phi.$$

Let $A^2 = a$, $B^2 = b$, $A B \cos\phi = c$, then

$$\cos^2\phi = \frac{c^2}{a b} = \left(\frac{c}{b}\right)^2 \div \frac{a}{b},$$

or

$$\cos\phi = \frac{c}{b} \frac{1}{\sqrt{\frac{a}{b}}};$$

and since

$$\frac{c}{b} = \frac{\gamma}{\beta} \cdot \frac{h_2}{h_3} = \frac{\gamma}{\beta} \cdot \frac{\beta_0}{\gamma_0},$$

and

$$\frac{a}{b} = \frac{\alpha}{\beta} \cdot \frac{h_2}{h_1} = \frac{\alpha}{\beta} \quad \frac{\beta_0}{\alpha_0},$$

we have therefore $\cos\phi$ given in terms of the dynamometer readings.

In the actual experiments it was found that

$$\frac{a}{b} = 1{\cdot}027\,\frac{\alpha}{\beta},$$

and

$$\frac{c}{b} = 1{\cdot}0475\,\frac{\gamma}{\beta}.$$

The apparatus being arranged as in Fig. 58, where T is the transformer and a', b', c' the three dynamometers, a series of observations was taken with varying resistances r in the external secondary circuit. In the annexed table the first column gives the number of the experiment, the second the number of rotations made by the dynamo per minute, each revolution producing eight inversions of the current; the third column gives the external secondary circuit resistance; the succeeding three columns give the dynamometer readings; the seventh and eighth give the ratios $a : b$ and $c : b$, calculated as above, being the ratios of $A^2 : B^2$ and $A\,B\cos\phi : B^2$, whilst the last column gives the calculated angle of lag between the primary and secondary currents, and is seen to approximate to 180deg. as the resistance of the secondary circuit is lessened.

No.	Speed of dynamo n.	Secondary circuit resistance r.	Dynamometer readings.			Current ratios.		Phase difference of primary and secondary currents. ϕ
			α	γ	β	$\frac{a}{b}$	$\frac{c}{b}$	
1	603	·409	126·2	121·9	126·4	1·026	1·010	175° 30′
2	606	1·953	129·6	118·3	119·2	1·117	1·039	169° 26′
3	605	3·224	99·8	84·9	84·4	1·215	1·054	162° 59′
4	600	4·462	118·9	91·9	89·6	1·363	1·074	156° 55′
5	601	5·713	82·8	56·3	54·5	1·561	1·082	150° 0′
6	607	6·958	116·5	70·3	67·1	1·784	1·097	145° 13′
7	609	8·542	93·1	47·0	44·4	2·152	1·109	139° 7′
8	601	9·760	102·5	44·4	41·0	2·569	1·134	135° 2′
9	610	11·000	87·9	33·7	30·9	2·923	1·142	131° 55′
10	606	12·240	98·8	32·4	29·2	3·476	1·162	128° 33′
11	602	13·470	97·6	26·3	23·6	4·249	1·167	124° 29′
12	602	14·790	104·0	24·7	21·6	4·969	1·203	122° 40′

Prof. Ferraris then proceeds to inquire how these experimental results agree with the deductions from the theory.

Taking the equations as on p. 302 for the two circuits, and putting i_1 and i_2 for the values at any instant t of the primary and secondary currents, and $e = E \sin pt$ as the value of the instantaneous impressed electromotive force on the primary,

we have
$$L\frac{di_1}{dt} + M\frac{di_2}{dt} + Ri_1 = e \quad . \quad . \quad . \quad . \quad (1)$$

and
$$N\frac{di_2}{dt} + M\frac{di_1}{dt} + Si_2 = 0 \quad . \quad . \quad . \quad . \quad (2)$$

and
$$e = E \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is easy to show by substitution that these two differential equations are satisfied by the solutions

$$i_1 = I_1 \sin (pt - \alpha) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$i_2 = I_2 \sin (pt - \beta) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

if, also, the ratio of the maximum of the primary current I_1, and the secondary current I_2, is taken as expressed by

$$\left(\frac{I_1}{I_2}\right)^2 = \frac{S^2 + p^2 N^2}{p^2 M^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and if
$$\tan (\beta - \alpha) = -\frac{S}{pN} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These formulæ we have proved on pp. 150 and 151, § 21, Chap. III.

If we write for shortness $pM = \mu$, $pN = \lambda$, and $\beta - \alpha = \phi$, we can express the above relations by

$$\left(\frac{I_1}{I_2}\right)^2 = \frac{\lambda^2 + S^2}{\mu^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\tan \phi = -\frac{S}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

But since ϕ is the angle of phase difference of the primary and secondary currents, and since $\cos \phi$ has been experimentally determined, we have the means of testing the truth of this theory. For since $\tan \phi = -\frac{S}{\lambda}$, it follows that

$$\left(\frac{I_2}{I_1}\right)^2 \cos^2 \phi = \left(\frac{\lambda}{\mu}\right)^2,$$

but with the previous notation we have seen that the ratio of the current values is obtained from the dynamometer readings, and is

$$\left(\frac{I_1}{I_2}\right)^2 = \frac{a}{b} \text{ (an observed quantity)},$$

whence

$$\cos^2\phi = \left(\frac{\lambda}{\mu}\right)^2 \frac{b}{a};$$

but we saw that $\cos^2\phi$ was measured by the value of $\left(\frac{c}{b}\right)^2 \frac{b}{a}$. Comparing these two values of $\cos^2\phi$, we have

$$\frac{c}{b} = \frac{\lambda}{\mu} = \frac{N}{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now $\frac{N}{M}$ is a constant quantity. It is the ratio of the self-induction of the secondary circuit to the mutual inductance of primary and secondary, and as the causes of variation of this ratio are in the iron, whatever varies the one varies the other. Hence, on this theory the ratio of $c : b$ should be constant. But on turning to the table on p. 307 we see that this ratio increases with the secondary circuit resistance, and is not constant. The explanation of this discrepancy is that in our simple theory we have taken no account of the lag of induction behind magnetising force due to hysteresis and Foucault or eddy currents in the iron. The differential equations are worked out on the tacit but erroneous assumption that the magnetisation of the iron is in consonance as regards phase with the magnetising force. Starting then on a fresh basis of theory, Prof. Ferraris shows that the differential coefficients by which we multiply the coefficients of induction in our fundamental equations must be modified to suit the fact that a delay or lag in the magnetisation of the iron takes place. If we write $\left(\frac{d i_1}{d t}\right)_{t-\theta}$ as a symbol for the value of the time rate of change of i_1, not at the instant t, but at an interval of time θ later, and similarly for i_2, we must write the fundamental equations thus:—

$$L\left(\frac{d i_1}{d t}\right)_{t-\theta} + M\left(\frac{d i_2}{d t}\right)_{t-\theta} + R i_1 = e \quad . \quad . \quad (11)$$

$$N\left(\frac{d i_2}{d t}\right)_{t-\theta} + M\left(\frac{d i_1}{d t}\right)_{t-\theta} + S i_2 = 0 \quad . \quad . \quad (12)$$

In these equations M, N, and L are approximately constants, but besides being determined by the geometrical form of the coils they are affected by the circumstances which cause the retardation θ. Hence they differ in value from the coefficients of induction under no supposed retardation of induction; we will call them the coefficients of apparent induction. With the above equations (11) and (12) we have, as before, the equation of impressed electromotive force $e = \mathrm{E} \sin p\,t$. Taking the form of the solutions for i_1 and i_2 as before,

$$i_1 = \mathrm{I}_1 \sin(p\,t - a), \qquad i_2 = \mathrm{I}_2 \sin p\,t - \beta,$$

we have for the value of the differential coefficients

$$\left(\frac{d\,i_1}{d\,t}\right)_{t-\theta} = p\,\mathrm{I}_1 \cos(p\,t - a - p\,\theta),$$

and $$\left(\frac{d\,i_2}{d\,t}\right)_{t-\theta} = p\,\mathrm{I}_2 \cos(p\,t - \beta - p\,\theta),$$

which give us the true rates of change of the currents at instants later than t by an interval θ. Substituting these in the equations (11) and (12), and using the same abbreviations μ and λ, Prof. Ferraris deduces two others, viz.:—

$$\mathrm{I}_1\,\mu \sin(a + p\,\theta) + \mathrm{I}_2\,[\lambda \sin(\beta + p\,\theta) + \mathrm{S} \cos \beta] = 0 \quad . \quad (13)$$

$$\mathrm{I}_1\,\mu \cos(a + p\,\theta) + \mathrm{I}_2\,[\lambda \cos(\beta + p\,\theta) - \mathrm{S} \sin \beta] = 0 \quad . \quad (14)$$

From these we have at once,

$$\mathrm{I}_1^2\,\mu^2 = \mathrm{I}_2^2\,[\lambda^2 + \mathrm{S}^2 + 2\,\lambda\,\mathrm{S} \sin p\,\theta]. \quad . \quad . \quad (15)$$

and putting $a - \beta = \phi =$ the lag between primary and secondary current, we have also from (13) and (14),

$$\mathrm{I}_1^2\,\mu^2 + \mathrm{I}_2^2\,\lambda^2 + 2\,\mathrm{I}_1\,\mathrm{I}_2\,\mu\,\lambda \cos \phi = \mathrm{I}_2^2\,\mathrm{S}^2. \quad . \quad (16)$$

and from (15) and (16) we have also

$$\mathrm{I}_2^2\,(\lambda + \mathrm{S} \sin p\,\theta) + \mathrm{I}_1\,\mathrm{I}_2\,\mu \cos \phi = 0 \quad . \quad . \quad (17)$$

In the equations (15), (16), and (17) we can introduce the magnitudes a, b, and c, which are deduced directly from the dynamometer readings for

$$a = \mathrm{I}_1^2; \; \mathrm{b} = \mathrm{I}_2^2; \; \cos \phi = \frac{c}{b}\,\frac{1}{\sqrt{\frac{a}{b}}}.$$

Hence we get

$$\frac{a}{b}\mu^2 = \lambda^2 + S^2 + 2\lambda S \sin p\theta \quad . \quad . \quad . \quad (18)$$

$$a\mu^2 + b\lambda^2 - 2c\mu\lambda = bS^2 \quad . \quad . \quad . \quad . \quad (19)$$

$$\frac{c}{b} = \frac{\lambda}{\mu} + \frac{S}{\mu}\sin p\theta \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This last equation (20) shows us that on the assumption of a retardation of induction the ratio $\frac{c}{b}$ is not constant, but increases with the secondary resistance S, and that if this ratio is plotted as ordinates to the several values of S, we find it to be a straight line sloping upwards. An inspection of the table on p. 307 shows this to be the case, for we see that the column of figures headed by $\frac{c}{b}$ steadily increases, and if these numbers are plotted off as ordinates to abscissæ, represented by the series of numbers in the column headed by r, we find that the ratio $c : b$ increases nearly proportionately to the secondary resistance r. Moreover, the equation (20) furnishes a means by which the time retardation of induction θ may be calculated for each value of secondary resistance when the ratios $\frac{b}{c}$ and $\frac{\lambda}{\mu}$ and $\frac{S}{\mu}$ are known.

Prof. Ferraris then points out that the causes of this retardation are to be looked for in hysteresis, and also in the existence of Foucault or eddy currents in the core even though this may consist of bundles of fine wire. He then proceeds to discuss a problem of three circuits 1, 2, and 3, each having coefficients of self-induction L_1, L_2, and L_3, and coefficients of mutual induction, viz., M_1 between 1 and 2, M_2 between 1 and 3, and M_3 between 2 and 3, and to solve the current equations so resulting, with the final result that he arrives at equations identical with (18), (19), and (20). This indicates that we might consider the reaction of the eddy currents in the iron as if it were the result of a current induced in a closed tertiary circuit wound on a core in which eddy currents were perfectly prevented. The general result of these eddy currents may be summed up in saying that they dissipate a fraction of the supplied energy and create a retardation in the phase of the magnetisation or magnetic induction in the iron core relatively to the magnetising force.

Prof. Ferraris next proceeds to show how this retardation interval can be calculated. The ratio $\frac{\mu}{\lambda}$ or $\frac{M}{N}$ in the case of a transformer with a closed iron core is simply the ratio of the number of wire turns on the primary and secondary coils, or

$$\frac{\mu}{\lambda}=\frac{M}{N}=\frac{N_1}{N_2}.$$

The ratio of the mutual to the secondary self-inductance is the ratio of the number of lines of force embraced by the primary and secondary circuits respectively. In the case of the open circuit transformers this relation is merely approximate, but in the case of the particular form of Gaulard and Gibbs transformer with which these experiments were made the ratio above mentioned was found to be a number close to unity, for the number of turns in the two circuits was equal.

Turning back, then, to equations (18), (19), and (20), we have from (20)

$$\sin p\,\theta = \left(\frac{c}{b} - \frac{\lambda}{\mu}\right)\frac{\mu}{S} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and from (19) we have

$$\mu = \frac{S}{\sqrt{\frac{a}{b} + \left(\frac{\lambda}{\mu}\right)^2 - 2\frac{\lambda}{\mu}\frac{c}{b}}} \quad . \quad . \quad . \quad (22)$$

Substituting the value of μ from (22) in (21) we can calculate $\sin p\,\theta$, and hence $p\,\theta$, or the angle of retardation for various values of S, the secondary resistance. In order to determine the magnetic retardation for various cores, experiments were made with five cores of iron in various forms.

No. 1 core was of iron wires, each ·65 millimetres diameter, bound round a rod of wood. The total diameter of the core was 46 millimetres and weight 2·92 kilogrammes.

No. 2 core was formed of thicker iron wires, each 3·3 millimetres diameter.

The inclusive diameter was 43 millimetres and weight 3·55 logrammes.

No. 3 core was formed of 16 bars of soft iron, each having section of 60 square millimetres. Total weight 3·44 kilo-ammes.

No. 4 core consisted of a drawn tube of soft iron 3 milli-etres thick and weighing 2·16 kilogrammes.

No. 5 core was a solid iron bar 42 millimetres diameter and eighing 6·59 kilogrammes.

This series of cores was selected to render more and more nsible the effect of eddy currents, and to see if the variation θ corresponded with the indications of theory.

A series of dynamometer observations was taken, and the served values of the ratio $\frac{c}{b}$ given at once from the current adings were tabulated against the various values of S, the condary circuit resistance.

In each case $c \div b$ was found to increase with S. The increase $c \div b$ was, however, much greater in the case of the solid and ibular core than in the case of the wire cores 1 and 2, thus dicating that the departure from constancy in this ratio creased in proportion as eddy currents had more opportunity be created. If values of S were plotted out horizontally as oscissæ, it was found that for the various cores the correspond-g values of $c \div b$ lay on approximately straight lines inclined pwards at steeper angles to the horizontal for the cores 5 and 4 ian for 1 and 2.

Then from the determined value of $\frac{\lambda}{\mu}$ (equal to about 1·0086), nd the observed values of $c \div b$, values were found by the help f equations (21) and (22) for sin $p\,\theta$, with the following sults:—

Core No. 1, Fine Wire. — As the secondary resistance in-reased from 1·953 to 14·790 units, sin $p\,\theta$ fell from 0·16 to ·10. Taking for sin $p\,\theta$ a mean value of 0·11, this corre-ponds to an angle of 6° 19′ of lag of magnetisation behind agnetising force. The number of revolutions of the dynamo as 604 per minute, and as each revolution gave four complete lternations, this gave 40·28 complete alternations per second s the frequency of the currents.

Core No. 2, Coarse Wire.—As the secondary resistance varied from 1·953 to 14·79 units, sin $p\,\theta$ varied from 0·29 to 0·18. This corresponds to a mean angle of lag of 10° 22′.

Core No. 3, Iron Rods.—Secondary resistance varying between the same limits, sin $p\,\theta$ varied from 0·39 to 0·44, and the mean value corresponded to an angle of lag of 24°.

Core No. 4, Iron Tube.—For the same range was found a mean angle of lag of 36°.

Core No. 5, Iron Bar.—Under the same circumstances the angle of lag was 43°.

Hence these experiments indicate that when eddy currents in the core have unrestricted play the magnetisation of the iron lags behind the magnetising force by nearly an eighth part of a complete period, but that with very well divided core it may amount to 5° or 6°.

In the Paper of Lord Rayleigh to which reference has been made (*Phil. Mag.*, December, 1886), an experiment is described showing the effect of the introduction into a helix of copper wire of a solid iron core and of a bundle of wires of similar iron of equal aggregate section and weight.

The experiments were made with the induction bridge described on p. 247, and the readings given are the differences of the values of the ohmic resistance and self-induction of the copper wire helix with and without the iron cores. In the first case one iron wire was taken 1·2 millimetre in diameter and compared with two bundles of 7 and 17 wires respectively of equal aggregate section. The helix of wire was wound on a glass tube, and had a length of 28·6 centimetres and 205 turns. The results were as follows :—

Increase.	Core of 1 wire.	Core of 7 wires.	Core of 17 wires.
Of resistance of copper wire circuit............	1·3	0·3	0·2
Of self-induction of copper wire circuit	13°	18°	18°

The results are in arbitrary units of bridge wire lengths and compensator coil deflection, but they show sufficiently that the single solid wire increases the resistance and also the self-induction of the primary circuit, but that when an equal mass of iron in a highly divided condition is employed the resistance of the primary circuit is only increased by a very small amount, and the self-induction is more increased than it is when the iron is entire. This demonstrates conclusively the existence of these mass-currents in the iron. A similar experiment with an iron wire 3·3 millimetres in diameter was made. The resistance and the self-induction of the helix were increased by 4·4 and 28½ deg. (arbitrary units) respectively by the introduction of this iron wire as a core. On employing a bundle of 35 soft iron wires of equal aggregate section the self-induction was increased by 84 deg. and the resistance by 1·6, both measured in the arbitrary units. These experiments illustrate well that if an iron core is employed not much divided then the effect of the eddy currents generated in it is to augment the resistance of the circuit more and the inductance less than if a core is provided in which these parasitic currents are prevented. Hence, in induction coils in which the object is the production of secondary electromotive force in a secondary circuit the end is best achieved by the insertion of an iron core sufficiently divided to quench the eddy currents but otherwise having the highest possible permeability. Experiment seems to show that there is a limit to the advantageous subdivision of the iron core, and that cores formed of excessively fine iron wires may be disadvantageous by failing to secure the highest permeability consistent with practical absence of eddy currents.

§ 19. **Coefficient of Mutual Inductance Influenced by Eddy Currents in the Core.**—The substitution of the observed quantities in the equation (22) giving the value of μ, viz.,

$$\mu = \frac{S}{\sqrt{\frac{a}{b} + \left(\frac{\lambda}{\mu}\right)^2 - 2\,\frac{\lambda}{\mu}\,\frac{c}{b}}},$$

gives us a value of the coefficient of mutual induction of the primary and secondary coils. For, recollecting that

$$M = \frac{T}{2\pi}\mu,$$

where M is the mutual inductance and T the periodic time, we have from the observed quantities the value of M. This value of M so determined may be called the *apparent coefficient of induction*, for it is not equal to the coefficient of induction M′ which would be found if eddy currents were entirely suppressed. The amount by which M′ differs from M will depend on the retardation of magnetisation—that is, on the strength of these parasitic currents. The eddy currents flowing in closed circuits have upon the primary current the effect which any other proximate current has, viz., they increase the primary resistance and reduce its inductance. The same also with regard to the secondary circuit. Whatever reduces, however, the self-inductions of those circuits reduces their mutual induction, because owing to the intertwining of the primary and secondary circuits the same sheaf of lines of force is embraced by both circuits. The direction of the eddy currents is such as to generate in the core an induction opposing that due to the resultant magnetising effect of the primary and secondary currents taken together. Hence, the net result is a less induction in the core and a less mutual induction than would be the case if the eddy currents were suppressed.

Prof. Ferraris gives a table showing the values of the coefficients of induction for the various cores and the comparison of these obtained according to the method of equation (22), and those obtained under circumstances equivalent to the annulment of eddy currents. A series of experiments was made with each of the five cores. A comparison was made between the "throw" of a ballistic galvanometer placed in circuit with the secondary when known currents produced in the primary by a constant battery were reversed, and the "throw" produced when a condenser charged at the terminals of the primary was discharged through the same galvanometer.

Let R be the resistance of the primary and S that of the secondary, including the galvanometer and leads, and C the capacity of condenser; then supposing that a potential difference of V was created at the ends of the primary by a constant battery, the current through the primary when steady would be $V \div R$, and on reversing this current suddenly a quantity of electricity equal to $2\, M' \dfrac{V}{RS}$ would be sent through the galvanometer. If the condenser be charged at the same points its contents would be C V units of quantity, and if the respective "throws" were α' and α, then we have

$$\frac{2\,M'\,V}{R\,S\,C\,V} = \frac{\sin\frac{\alpha'}{2}}{\sin\frac{\alpha}{2}},$$

or

$$M' = \frac{1}{2}\,R\,S\,C\,\frac{\sin\frac{\alpha'}{2}}{\sin\frac{\alpha}{2}}.$$

The values of M′ so obtained are given in the annexed table in comparison with the values of M arrived at by the experiments with alternating currents.

Nature of Cores.	$\frac{2\pi}{T}$ M, or μ.	Apparent Coefficient of Induction = M.	Real Coefficient of Induction = M′.
Iron wires, fine ...	9·09	·0359	·0358
Iron wires, coarse.	10·10	·0401	·0446
Iron rods............	7·58	·0301	·0538
Iron tube	4·90	·0194	·0480
Solid iron bar......	4·54	·0180	·0546

The column headed μ is the values of $\frac{2\pi}{\tau}$ M obtained according to equation (22), p. 312, where S had a value of 5·712 ohms and the frequency $\frac{1}{\tau}$ a value of 40 per second. It will be seen that M, the coefficient of apparent mutual induction, in presence of the eddy currents at first increases and then decreases as the subdivision of the core grows less. The first increase is no doubt due to an increase in the permeability of the core by using coarser iron wires, but as soon as the core ceases to be divided and eddy currents have free course we note a remarkable diminution in the coefficient of mutual induction. On the other hand, the column M′, which gives the coefficient without disturbance by eddy currents, shows a slight increase at first, but remains tolerably constant for the four last cores. The values for M′ as they are in Prof. Ferraris's memoir are only half those given in the foregoing table, but this is obviously a slip of the pen. It is clear that M and M′ must be equal in value when the subdivision of the core is carried to that point at which eddy currents have no great importance, and from the values of M′ we see in a marked manner that in the finely-divided cores M and M′ have the same values, but that in proportion as eddy currents increase in strength the value of M falls below that of M′.

§ **20. Dissipation of Energy in Iron Cores.**—If at any instant the impressed electromotive force at the terminals of the primary circuit has a value e, and the current a value i_1, then in an element of time dt the work spent on the transformer is $e\,i_1\,dt$. If R and S be the ohmic resistances of the two circuits, then $(\mathrm{R}\,i_1^2+\mathrm{S}\,i_2^2)\,dt$ represents the work transformed into heat in the primary and secondary circuits respectively, including, of course, the work performed internally as well as externally in the transformer secondary circuit. Then the energy dissipated in the time dt is

$$(e\,i_1-\mathrm{R}\,i_1^2-\mathrm{S}\,i_2^2)\,dt,$$

and the mean energy dissipated during one complete period of duration T is

$$\mathrm{P}=\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}(e\,i_1-\mathrm{R}\,i_1^2-\mathrm{S}\,i_2^2)\,dt \quad . \quad . \quad . \quad (1)$$

From equations (11) and (12), p. 309, we have

$$e\,i_1-\mathrm{R}\,i_1^2=\mathrm{M}\,i_1\left(\frac{d\,i_2}{d\,t}\right)_{t-\theta}+\mathrm{L}\,i_1\left(\frac{d\,i_1}{d\,t}\right)_{t-\theta},$$

and
$$-\mathrm{S}\,i_2^2=\mathrm{M}\,i_2\left(\frac{d\,i_1}{d\,t}\right)_{t-\theta}+\mathrm{N}\,i_2\left(\frac{d\,i_2}{d\,t}\right)_{t-\theta};$$

hence, if we add these two equations, and substitute in (1) the values of i_1, i_2, $\left(\frac{d\,i_1}{d\,t}\right)_{t-\theta}$ and $\left(\frac{d\,i_2}{d\,t}\right)_{t-\theta}$ viz. :—

$$i_1=\mathrm{I}_1\sin(p\,t-\alpha),\quad i_2=\mathrm{I}_2\sin(p\,t-\beta),$$

$$\left(\frac{d\,i_1}{d\,t}\right)_{t-\theta}=p\,\mathrm{I}_1\cos(p\,t-\alpha-p\,\theta),$$

$$\left(\frac{d\,i_2}{d\,t}\right)_{t-\theta}=p\,\mathrm{I}_2\cos(p\,t-\beta-p\,\theta),$$

we get the following:—

$$P=\frac{\pi}{T}\left[I_1^2 L+I_2^2 N+2 I_1 I_2 M \cos(\beta-a)\right]\sin p\,\theta,$$

or, $$P=\frac{\pi}{T}\left[I_1^2 L N+I_2^2 N^2+2 I_1 I_2 N M \cos(\beta-a)\right]\frac{1}{N}\sin p\,\theta.$$

Now if N_1 and N_2 are the numbers of turns of the primary and secondary circuits, since the same magnetic flux of induction traverses both circuits, we can write

$$L=k N_1^2, \qquad N=k N_2^2, \qquad M=k N_1 N_2,$$

where k is the same constant. Thus $L N=M^2$, and putting as before,

$$p M=\mu, \qquad p N=\lambda, \qquad \beta-a=\phi,$$

we find as an equivalent for the expression for P the final result,

$$P=(I_1^2 \mu^2+I_2^2 \lambda^2+2 I_1 I_2 \mu \lambda \cos\phi)\frac{1}{2\lambda}\sin p\,\theta.$$

Comparing this last with equation (16) on p. 310, we see that

$$I_1^2 \mu^2+I_2^2 \lambda^2+2 I_1 I_2 \mu \lambda \cos\phi=I_2^2 S^2.$$

Hence we have finally $$P=\frac{I_2^2 S^2}{2\lambda}\sin p\,\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) is an expression for the mean energy dissipated in a complete period, but the mean value of the energy converted into heat in the secondary circuit in a complete period is

$$Q=\frac{I_2^2 S}{2}.$$

Therefore $$\frac{P}{Q}=\frac{S}{\lambda}\sin p\,\theta \quad . \quad . \quad . \quad . \quad (3)$$

Combining this last with equation (20) on p. 311, we have

$$\frac{P}{Q}=\frac{\mu}{\lambda}\left(\frac{c}{b}-\frac{\lambda}{\mu}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This remarkably simple expression gives us in observable quantities the value of the ratio between the mean value of the energy dissipated in a complete period in the core and that dissipated in the same time in the secondary circuit.

Taking a particular value for S, viz., 6 ohms, Prof. Ferraris found for the five kinds of core above described values of $\frac{P}{Q}$ equal respectively to

$$0{\cdot}07, \quad 0{\cdot}11, \quad 0{\cdot}30, \quad 0{\cdot}72, \quad 0{\cdot}90.$$

The mean energy dissipated in a period in the primary circuit is $\frac{R\,I_1^2}{2}$, and that in the secondary is $\frac{S\,I_2^2}{2}$, I_1 and I_2 being the maximum values of the currents; and since this last expression has been denoted by Q, and the energy dissipated in the core by P, we have for the total energy supplied to the transformer

$$Q + R\frac{I_1^2}{2} + P,$$

and for the ratio of the energy transformed in the secondary circuit to this the expression

$$\epsilon_1 = \frac{Q}{Q + R\frac{I_1^2}{2} + P},$$

or

$$\epsilon_1 = \frac{S}{S + R\left(\frac{I_1}{I_2}\right)^2 + S\frac{P}{Q}}.$$

This quantity ϵ_1 is the *total electrical efficiency* of the transformer. If ρ denote the resistance of the secondary circuit inside the transformer coils, then the quantity ϵ_2, where

$$\epsilon_2 = \frac{S - \rho}{S + R\left(\frac{I_1}{I_2}\right)^2 + S\frac{P}{Q}},$$

may be called the *external efficiency*, because it represents the fraction of the total supplied energy which is available in the external part of the secondary circuit.

Now since $\left(\frac{I_1}{I_2}\right)^2 = \frac{a}{b}$, and we have seen that

$$\frac{P}{Q} = \frac{\mu}{\lambda}\left(\frac{c}{b} - \frac{\lambda}{\mu}\right),$$

we have by substitution in the expression for ϵ_1 and ϵ_2 the results

$$\epsilon_1 = \frac{S}{R\frac{a}{b} + S\frac{\mu}{\lambda}\frac{c}{b}} \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\epsilon_2 = \frac{S - \rho}{R\frac{a}{b} + S\frac{\mu}{\lambda}\frac{c}{b}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which ϵ_1 and ϵ_2 are given in terms of the observable dynamometer readings. As in closed iron circuit transformers the ratio $\frac{\mu}{\lambda}$ is unity, and nearly so also in the case of the open circuit Gaulard and Gibbs transformer experimented upon, we can reduce these expressions finally to

$$\epsilon_1 = \frac{S}{R\frac{a}{b} + S\frac{c}{b}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$\epsilon_2 = \frac{S - \rho}{R\frac{a}{b} + S\frac{c}{b}}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We have then in the tabulated results of observation given on p. 307 all the data for calculating the efficiency for the various values of the secondary circuit resistance. In the transformer experimented upon the primary coil had a resistance of ·276 ohms and the secondary coil a resistance of ·285 ohms. Hence

$$R = \cdot 276, \qquad \rho = \cdot 285,$$

and the S of the equations above is the sum of ρ and the tabulated value of r or the external secondary resistance. Accordingly, taking from the table the values

$$r = \cdot 409, \qquad \frac{a}{b} = 1\cdot 026, \qquad \frac{c}{b} = 1\cdot 01,$$

$$R = \cdot 276, \qquad S = \cdot 285 + \cdot 409,$$

we have $$\epsilon_1 = \frac{\cdot 409 + \cdot 285}{\cdot 276 \times 1\cdot 026 + 1\cdot 01 \times (\cdot 409 + \cdot 285)}$$

$$= \frac{694}{784} = 71 \text{ per cent.}$$

Calculating in a similar manner the value of ϵ_1 for the various values of r, we have for the results as follows :—

External Resistance of Secondary $= r$.	Total Efficiency $= \epsilon_1$.	External Resistance of Secondary $= r$.	Total Efficiency $= \epsilon_1$.
·409	·71	8·542	·85
1·953	·85	9·760	·83
3·224	·875	11·000	·825
4·462	·87	12·24	·80
5·713	·876	13·47	·79
6·958	·859	14·79	·77

We see, therefore, that there is a *certain value of the resistance of the secondary circuit for which the efficiency is a maximum*, and that in this case it is about 5·7 ohms. There is a certain falling off in efficiency both when the transformer is very lightly and when it is very heavily loaded. Corresponding to this most advantageous resistance the total efficiency is about 87 per cent. and the external efficiency ϵ_2 about 82·5 per cent., and also for this load the magnetic lag is an angle whose sine is equal to ·12 or $\sin p\theta = \cdot 12$.

It is possible, therefore, from the results of the dynamometer observations to draw a curve of total efficiency which will show at once for what secondary resistance this is a maximum. Looking at the matter in a common-sense point of way, we see that since the energy transformed into heat or light in the secondary circuit is $\frac{1}{2} I_2^2 R_2$, if the secondary circuit had no resistance there would be no work done in it, and hence the transformer would have no efficiency. If the secondary circuit has an infinite resistance then there is no finite current generated in it, and hence no work done in it. Hence, for the two extremes of $R_2 = 0$ and $R_2 = \infty$ the transformer has no efficiency. Somewhere between these extremes it has a maximum efficiency, which may be 80 or 90 per cent. or more, and the curve of transformer efficiency plotted in terms of secondary circuit resistance is a curve which rises up to a maximum and then falls again to the datum line. The practical moral of this is that when a transformer is very heavily loaded—in fact, overloaded—its electrical efficiency falls again somewhat below its maximum value, and there is a certain range of secondary resistance within which the transformer is worked at its greatest advantage as a transforming device.

Prof. Ferraris has treated this part of the subject analytically instead of graphically, as we have done, and proceeded as follows:—

Taking the expression for the total electrical efficiency given on p. 320, viz.:—

$$\epsilon_1 = \frac{S}{S + R\left(\frac{I_1}{I_2}\right)^2 + S\frac{P}{Q}},$$

he introduces the value of $\left(\frac{I_1}{I_2}\right)^2$ —that is, of $\frac{a}{b}$ given in the equation (18) on p. 311—for

$$\left(\frac{I_1}{I_2}\right)^2 = \frac{a}{b} = \left(\frac{\lambda}{\mu}\right)^2 + \frac{S^2}{\mu^2} + 2\frac{\lambda S}{\mu^2}\sin p\,\theta,$$

and also $\frac{P}{Q} = \frac{S}{\lambda}\sin p\,\theta$ (equation 3, p. 319). Hence

$$\epsilon_1 = \frac{S}{S\left(1 + 2\frac{R\lambda}{\mu^2}\sin p\,\theta\right) + R\left(\frac{\lambda}{\mu}\right)^2 + S^2\left(\frac{R}{\mu^2} + \frac{1}{\lambda}\sin p\,\theta\right)}.$$

The quantities μ, λ, and θ are really functions of S, but if for a first approximation we take them as constant, we can on this assumption find the value of S, for which ϵ_1 becomes a maximum. Differentiating the above with respect to S, and equating to zero, we find the value S_1, for which ϵ is a maximum, to be

$$S_1 = \frac{\lambda}{\sqrt{1 + \frac{\mu^2}{R\lambda}\sin p\,\theta}}.$$

To apply this to the actual case in question, we take $\frac{\mu}{\lambda} = 1$ as before, and taking from the table on p. 317 the mean value of μ as about 9, and the value of $p\,\theta$, or the angle of magnetic lag, as about 6 deg. for the finely divided core, we have $\sin p\,\theta = \cdot 12$.

Now, $R = \cdot 276$ ohms, $\mu = 9$, $\lambda = \mu = 9$, $\sin p\,\theta = \cdot 12$.

Hence $$S_1 = \frac{9}{\sqrt{1 + \frac{9}{\cdot 276} \cdot 12}} = 4 \cdot 1 \text{ ohms.}$$

Corresponding to this value of S we have

$$\epsilon_1 = \cdot 87 \quad \text{and} \quad \epsilon_2 = \cdot 825.$$

These are approximately the total and the external maximum efficiencies at this load.

If the eddy currents be supposed to be entirely suppressed so that the magnetic lag $p\,\theta$ is supposed to be zero, then corresponding to this it can be shown that we should have

$$\epsilon_1 = \cdot 93 \quad \text{and} \quad \epsilon_2 = \cdot 89.$$

In other words, the loss of energy due to the dissipation in the iron core is some 6 per cent. at the maximum efficiency.

We shall defer until a later chapter the consideration of a number of other problems in the theory of transformers with iron cores.

CHAPTER V.

DYNAMICAL THEORY OF CURRENT INDUCTION.

§ 1. **Electro-Magnetic Theory—Electric Displacement.**—In the matter so far before the reader attention has been directed either to the description of phenomena or else the elementary theory presented has been based upon the evident fact that electro-magnetic induction can take place across space without any inquiry into any possible mechanism by which this may be effected. Attention may at this stage be directed to some modern views of the induction phenomena which have been the outcome of Maxwell's study of Faraday's work. The cardinal principle of these methods of viewing the subject is the denial of action at a distance. That is to say, if at any point in a field we find lines of force originated by a current flowing in some conductor, these lines of force did not suddenly appear there without anything happening in the interspace, but have been the result of a propagation of electro-magnetic effects through space, and the apparent results, whether appearance of a magnetic field or the generation of an electric current in a conductor, are the consequence of successive changes in closely contiguous places, and not the result of operations at a distance without intermediate machinery. Whenever we find an electro-magnetic effect taking place at any locality we are directed therefore by these notions to look for its antecedents or consequences at the adjacent places, and the apparent phenomenon is not to be regarded as the whole of it, but to be taken as a portion of the whole of the effects which are produced in every part of the region or medium. The finite velocity of light, and the impossibility of accounting for its propagation on any other hypotheses than that of actual transmission of something across space, or the propagation of a state of stress and strain through a medium, led to attempts to settle between the rival hypotheses by crucial experiment, with the result that the vast bulk of accumulated evidence decides in favour of the existence in space of a *medium* which is not ordinary atomic matter, but which has the property, like ordinary material substance, of being the recipient or vehicle of energy, and possesses a quality of elasticity in that a *displacement* of one portion relatively to another calls forth a stress resisting that change. The term displacement here used must, however, be taken in its utmost generality, and not limited to the notion of a sliding or shearing of one part over an adjacent part, or an approximation of portions of a discrete or atomically constituted *ether*. The study of the phenomena of light indicates that along the path of a ray there are certain *changes* which are periodically repeated, such that at portions of the medium separated by a distance called a wave length changes of a similar kind are being coincidently effected. It would be too much to say that with our present knowledge we know what is the real nature of these changes. The application of mathematical analysis to optical phenomena has, however, led to the conclusion that we can offer a tolerably satisfactory account of them by making the supposition that there exists such a universally diffused *ether* or medium, the parts of which can change position relatively to each other, and that these changes of form are resisted by internal shearing stresses or elastic forces. From this point of view, now sometimes called the elastic solid theory of light, we may picture this ether to ourselves as a distortable but incompressible jelly-like solid, which exists everywhere and penetrates into the interior of all material bodies. Maxwell was led to the conclusion that electric and electro-magnetic phenomena might be explained by the supposition of an *electric medium* capable of certain internal motions and possessing certain mechanical properties, and to avoid the unscientific process of thought of postulating two different *ethers* was led to suppose that the medium on which electric effects and optical phenomena depend for their existence are one and the same.

One important element in Maxwell's electric theory is his conception of *electric displacement*. When an electrostatic force acts upon any part of a dielectric which is uniform and non-crystalline it is assumed that at all points along the line of electrostatic induction there is an *electric displacement* of electricity. The theory does not tell us what is the physical nature of this displacement. We may, in the first place, merely for the purpose of illustration, suppose that the unknown something which we call electricity is moved along a line of induction, and that on the removal of the electric force it returns to its original position, and that a dielectric or insulator is a material in which the electricity, when displaced by the application of an electrostatic stress or force, resists this displacement in virtue of an *electric elasticity*. The apparent charge on conductors, according to this view, is the electricity displaced out of, or into, the dielectric, and positive charge or electrification may be regarded as the possession of an excess which is extruded from the dielectric on to the conductor, and negative as a deficit when the conductor gives up some to the dielectric.

Maxwell's next principle is that change in electric displacement *is* an electric current whilst the change lasts. He calls this a *displacement current*, to distinguish it from a current in conductors called the *conduction current*. The displacement current is supposed to have, however, all the properties of an electric current. Conducting bodies must be regarded as those in which there is no elastic force resisting displacement, or, in other words, have no electric elasticity, and in which, therefore, electric displacement can go on continuously. The existence of a current of conduction is recognised by two co-existing effects—first, the dissipation of energy into heat; and second, the existence of magnetic force the direction of which is along closed lines described around the line of the current. The displacement current in dielectrics, which takes place at the instant of applying or changing the electric force, is also considered to be accompanied by magnetic force. In other words, we must consider the displacement current which takes place in a dielectric when electrostatic force acts on it as a very brief conduction current, and as originating a system of lines of magnetic induction—surrounding it, just as a conducting wire is so surrounded, by its loops or closed lines of magnetic induction. Conversely, when lines of magnetic force penetrate through an insulator or dielectric, any change in the density of those lines creates eddy displacement currents in the mass. If the lines penetrate through a conductor they produce, under similar circumstances, eddy currents of conduction, whose energy is ultimately frittered down into heat. Also, if a conductor is moved across a magnetic field so that it "cuts" lines of induction we have seen that if the conductor is a portion of a closed circuit it has a current of conduction produced in it. Similarly, if a dielectric body is moved in a magnetic field in a like manner it has during the continuance of the motion a displacement current produced in it. Since a dielectric circuit is always a closed circuit, a displacement current, or the production of electric displacement in it, is always the result of any change in the magnetic field in its interior. For the purpose of obtaining a rough illustrative working model of the actions going on in a dielectric submitted to the action of electric force, it is necessary to fall back on some *material* hypothesis of electricity—that is, we must conceive of electricity as a *something* which can be displaced relatively to the molecules of the dielectric, and that it resists this displacement, and that when this displacement is made under the action of electric stress the removal of this stress causes a disappearance of the displacement. Dr. Lodge has suggested a form of apparatus which can serve as a rough working model of this dielectric action, in which buttons sliding along a rod, and held in certain positions by elastic strings, represent the electric particles capable of elastic displacement.*

We may quantitatively define *electric displacement* by saying that in a homogeneous non-crystalline dielectric, if a plane be drawn perpendicular to the line of action of the resultant electric force, then under the operation of this electric force the quantity of electricity displaced normally across a unit of area of this plane is called the *electric displacement.* This displacement is of the nature of an elastic strain, and is removed when the electric force is removed. Let us fix our ideas by imagining a sphere immersed in a dielectric medium to receive an electric charge of quantity Q. Suppose this sphere to be sur-

* See Dr. O. J. Lodge "On a Model Illustrating Mechanically the Passage of Electricity through Metals and Dielectrics," *Phil. Mag.*, November, 1876.

rounded by a concentric spherical shell (Fig. 1) also immersed in the dielectric. On giving the central sphere a charge $+Q$ we know that on the inside surface of the insulated concentric shell will appear an inductive charge $-Q$ of equal quantity and opposite sign, and a charge $+Q$ on the outside surface. Let this spherical shell be very thin and be placed at a distance r from the central sphere, supposed to be very small. The electric force due to the central charge Q at the surface of the concentric shell is represented by $\frac{Q}{Kr^2}$, and this force exerts a displacing action on the electricity of the shell, causing positive electricity to be displaced outwards or in the direction of

Fig. 1.

the force and negative electricity to be displaced inwards or against the force.

The quantity K which appears in the above expression for the magnitude of the electric force is called the *dielectric constant*, or the *specific inductive capacity* according to Faraday, of the medium. If the dielectric is air or other gas K is very nearly unity, and the law of the force becomes the ordinary Newtonian law, viz., *force* varies as *quantity* divided by *square of distance*—that is, the electric force at any point due to a small quantity, Q, collected on a sphere is numerically equal to $\frac{Q}{r^2}$, where r is the distance of the point from the centre of the sphere. The quantity K is only truly unity for vacuum, and varies for known dielectrics from a little above unity for dry air up to a value of 6 to 10 for glass. In the case of metals and conducting bodies we may consider K to be infinitely great, and generally K is a number which expresses the ratio of the displacement in the given dielectric to the displacement which would take place under the same electric force if the dielectric was removed and a vacuum left in its place. The whole quantity displaced outwards through the conducting shell is $+ Q$, and since the radius of this spherical shell is r, its surface is $4 \pi r^2$ and the quantity displaced through unit of area of this shell in the direction of the force is $\frac{Q}{4 \pi r^2}$. This quantity, then, Maxwell calls *the electric displacement*, and denotes by the symbol **D**.

The electric force or resultant electric intensity at all points over the spherical shell is $\frac{Q}{K r^2}$, and this quantity Maxwell calls the *electromotive intensity* at that point and denotes it by **E**. We may also speak of **D** as the electric strain and **E** as the electric stress by an extension of usual mechanical language. The quotient of a *stress* by its corresponding *strain* is, in mechanics, called the *coefficient of elasticity* corresponding to that stress. For instance, the quotient of stretching force by longitudinal extension in the case of solid rods subjected to extending forces is called Young's Modulus of Elasticity, or the longitudinal elasticity. By a similar use of language the quotient electric stress by electric strain may be called the *electric elasticity*, and we have

$$\frac{\dfrac{Q}{K r^2}}{\dfrac{Q}{4 \pi r^2}} = \frac{4 \pi}{K} = \textit{the electric elasticity.}$$

Hence the series of numbers obtained by dividing the number $4 \times 3{\cdot}1415$ by the specific inductive capacities give a series of numbers which are the *electric elasticities* of these substances.

§ 2. **Displacement Currents and Displacement Waves.**—Maxwell's second fundamental conception, as we have mentioned, is that a displacement of electricity *whilst it is* taking place is an electric current. That is to say, the variation of displacement, whether of increase or decrease, is a movement of electricity which is in effect an electric current. A dielectric must, however, be considered as a body which does not permit of any but a very transient electric current passing through it. If continuous electric force is applied to it the dielectric is soon strained to its utmost extent, and no more current or flow or displacement takes place on it until the sign or direction of the electric force is reversed. A dielectric may be considered to be pervious to very rapidly reversed periodic currents, but very opaque or impervious to continuous currents. This is familiarly illustrated by the fact that a condenser inserted in a telephonic circuit does not stop telephonic communication, but does stop continuous currents. If $\mathbf{D}$ be at any instant the displacement at any point in a dielectric, then if $\mathbf{D}$ varies with the time so that $\frac{d\mathbf{D}}{dt}$ is its time rate of variation, then $\frac{d\mathbf{D}}{dt}$, or as it may be best written in Newtonian fashion $\dot{\mathbf{D}}$, is the *displacement current*, or rate of change of displacement. If at any point in a dielectric rapid changes of displacement are taking place, these variations of electric displacement are in effect electric currents producing magnetic induction in the surrounding portions of the dielectric. When we come to review certain recent experiments of Dr. Hertz we shall see that this view receives support from experimental research. An electric displacement taking place all along a certain plane is equivalent to a *current sheet*, and an electric displacement taking place along a certain line is a linear current. Electrostatically speaking, lines of electric displacement are lines of electrostatic induction, and these lines, when the displacement is changing, become lines along which electric current flow is taking place. The denial of action at a distance involves the assumption that the only portions of a dielectric which can act directly upon each other are those which are in immediate contact or are contiguous.

§ 3. **Maxwell's Theory of Molecular Vortices.**—The question then arises, by what mechanism can this be brought about? Given a medium possessing certain mechanical qualities, such as elasticity, a definite density, and capability of relative displacement of its parts, is it possible to imagine a construction which will mechanically account for the effects we have produced in electrical phenomena? Maxwell has replied to this inquiry by presenting his theory of *Molecular Vortices* which was put forward in the *Philosophical Magazine* for 1861 and 1862. A general account of these conceptions of Maxwell has been given in the "Life of James Clerk Maxwell," and as the limits of such an elementary treatise as the present one preclude any detailed account of the mathematical portion of this theory, we shall borrow the language of the authors of the above-mentioned work* in describing it. Maxwell supposes that any medium which can serve as the vehicle of magnetic force consists of a vast number of very small bodies called *cells*, capable of rotation, which we may consider to be spherical, or nearly so, when in their normal position. When magnetic force is transmitted by the medium or acts through it, these *cells* are supposed to be set in rotation with a velocity proportional to the intensity of the magnetic force, and the direction of rotation is related to the direction of the force in the same manner as the *twist* and *thrust* of a right-handed screw. We have thus all the magnetic field filled with *molecular vortices*, as Maxwell calls them, all rotating round the lines of force as axes. These cells as they revolve tend to flatten out like revolving spheres of fluid, and to become oblate spheroids; they thus contract along the lines of force and expand at right angles, creating a *tension* along the lines of force, and a *pressure* at right angles to them. These cells are supposed to be elastic spheres closely packed together and incapable of separating from each other. If any line of cells is set rotating the contraction of each cell along its axis of revolution must set up a tension or pull along that line, it behaves like a filament of muscular tissue, and contracts in length and swells out or increases in thickness. If several adjacent lines of cells or vortices are all set revolving in the same direction, the swelling out of each line causes them to press on each other; hence there is a lateral pressure and a

* "Life of James Clerk Maxwell." By Lewis Campbell and William Garnett. 1st Edition. 1882.

The general description of Maxwell's views in the above-mentioned work is due to Prof. Garnett, and in the annexed paragraphs the expository account of this theory is taken in part almost verbatim from the pages of this book.

longitudinal tension. In any space filled with these cells so revolving the lines of tension or axes of revolution of the cells will take up certain positions, depending on the necessity for the stresses to adjust themselves to equilibrium, and Maxwell has shown mathematically that such a system of lines of tension and correlated pressure is a force system which will distribute itself in a manner similar to that in which we find actual lines of magnetic force to do, and that the behaviour of magnetic poles to each other can be explained fully by the assumption of a tendency on the part of the lines of force between them to contract like elastic threads along their length, and to push one another apart when laid parallel and proceeding in the same direction. To account for the transmission of rotation from one cell to another in *the same direction*, and from one line of cells or vortex to the next, Maxwell supposed that there exists between the cells a number of extremely minute spherical bodies which can roll without sliding in contact with the vortex cells. These bodies serve the same purpose as "idle wheels" in machinery, which coming between a driving wheel and a following wheel serve to cause both to turn in the same direction. These minute spheres Maxwell supposed to constitute *electricity*. We shall speak of them collectively as the electric matter. These electric particles are furthermore supposed to be free to move in conductors; but in dielectrics they are tethered to one spot, or rather into one molecule of the substance, and can only be displaced a little way against an elastic resilience, which brings them back to their original position when the displacing force is withdrawn. Furthermore, we must assume that both cells and particles are very small, compared with the molecules of matter. The passage of electric particles from molecule to molecule in conductors, however, sets up molecular vibration, or generates heat. Something of the nature of friction must, therefore, be also postulated to account for the fact that the electric particle, when set moving in a conductor, gives up energy to the molecules, and the energy is in them dissipated in the form of heat. That there is some kind of rotation going on along the lines of magnetic force has been held by Maxwell to be indicated by the behaviour of a ray of polarised light when passing through a dielectric along a line of magnetic force, and he states* that Faraday's discovery of the magnetic rotation of the plane of polarised light furnishes complete dynamical evidence that wherever magnetic force exists there is matter, small portions of which are rotating about axes parallel to the direction of that force. The further assumption is made that the *cells* are composed of an elastic material, and that they can be distorted or squeezed slightly, returning again in virtue of their resistance to their original form. In order to obtain a clear mechanical conception of the inter-relation of the idle wheels or the electric particles and the revolving cells or lines of induction, let us figure to ourselves one element of the mechanism as it is supposed to exist in the dielectric.

Consider A and B (*see* Fig. 2) to be two wheels of india-

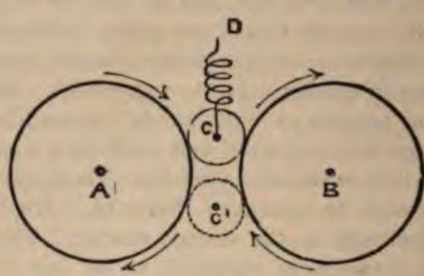


Fig. 2.

rubber, and that C is another small wheel lying between A and B and transmitting motion from one to the other. Let C be tethered to a fixed point, D, by an elastic spring, and let C be at the same time capable of rotation round its centre. Suppose A is set in rotation, clock-hand wise, whilst B is held fast, and that the wheel C cannot slip on A, the result will be to drag down C to the position of C^1, stretching the spring and displacing C. Let B be then set free; the wheel C continues to roll on A, and transmits its rotation to B. Owing to the assumed elasticity of the discs A and B, the wheel C can be drawn down between them, and yet within the limits of its displacement equally transmit the rotation of A to B without slip. This same action

* Article "Faraday," *Encyclopædia Britannica*, 9th edition.

of a preliminary displacement of C and subsequent rotation of B will take place if the wheel B possesses *inertia*—that is, if we assume it to be a heavy wheel which cannot in virtue of its mass be set rolling with finite speed in an infinitely short time.

If, then, we suppose a long row of such wheels with intermediate displaceable idle wheels, the main wheels being heavy bodies, the result of causing the first wheel to rotate would be to propagate along the line a successive displacement of the idle wheels, and to set the main wheels successively in rotation. Translating these mechanical concepts into their electrical equivalents Maxwell considers that the heavy wheels are the analogues of the molecular vortices or lines of force, and that their density is determined by what we call the magnetic permeability of the medium; the elastically displaceable idle wheels are the electricity in the dielectric; and that when a line of force is brought into existence in a dielectric, or, in other words, when a line of cells is set rotating, this action propagates itself outwards, producing successive displacements of the electric particles, or generates a displacement wave, and is accompanied by the successive appearance of rotation in the cells, or by the propagation of a wave of electromagnetic force.

The velocity of propagation of this wave will depend on the elastic forces restraining displacement, and on the inertia of the revolving vortices. We have seen that the elasticity of the dielectric is expressed by the quantity $\frac{4\pi}{K}$, where K is the specific inductive capacity. We shall see later on that the electromagnetic density of the medium is expressed by $4\pi\mu$, where μ is the magnetic permeability.

The velocity of propagation of a disturbance through an elastic medium is numerically equal to the quotient of the square root of its effective elasticity e, by the square root of its density d, or by $v=\sqrt{\frac{e}{d}}$.

If, then, for the electromagnetic medium $e=\frac{4\pi}{K}$ and $d=4\pi\mu$ we have $v=\frac{1}{\sqrt{K\mu}}$, or the velocity of lateral propagation of a wave of electric displacement or of magnetic force in a medium is numerically equal to the square root of the reciprocal of the product of its specific inductive capacity and its magnetic permeability. Such a mechanical hypothesis shows us how the spin of one line of vortices results in producing displacement of the idle wheels or electricity along lines which are circles described round the initial vortex as axis, and in propagating outwards the vortex spin or magnetic force with a finite velocity from one line of molecular vortices to another.

By the aid of the ideas which were discussed in the last section we are enabled to arrive at a mechanical conception which helps us to connect together observed facts, and which, even if not a real representation of what is taking place, is at least a working model which may not be an altogether incorrect imagery of the actions taking place when an electric current is started in a wire.

An electric current on this hypothesis is a flow or progression of the electric particles which are free to move forward in a conductor, and which only can move steadily forward, owing to their incompressibility, when the circuit in which they flow is a complete circuit. Suppose a thin conductor bent into the form of a very large circle, and that an electromotive force urges a procession of electric particles round it. As these particles go forward they cause the electric *cells* next them to rotate, and the motion of this line of cells embracing the line of current will be just like that which would take place if a bracelet of spherical beads strung on an elastic thread were rolled along a round rod which it closely embraces. Each bead would turn over and over, rolling on the rod, and the motion of the whole bracelet would be like that of a tightly-fitting india-rubber umbrella ring pushed along a round ruler. The progression of the electric particles would start circular vortex rings revolving round the line of motion. This corresponds to the fact that a linear current creates a magnetic field composed of circular embracing lines of forces. The first or adjacent line of vortices would, by the intervention of the idle wheels, set in rotation another set of cells lying on a concentric line, and cause them to rotate in the same manner as the first ones. Also, it would cause a backward displacement of the intermediate idle wheels, if we consider that only the central line of electric particles are conducting matter, and that the next and all succeeding rows are in a dielectric. The starting of the progressive movement of the line of electric particles in the conductor will result in an elastic displacement in the *opposite* direction of all surrounding electric particles in the dielectric along lines parallel to the line of current; and also in setting up a system of molecular vortices composed of revolving cells, the axes of these vortices being coaxial circles described round the line of flow, the rotations and displacements being propagated out laterally from the line of current. In consequence of the fact that the revolving cells are supposed to possess inertia or mass, and that all the mechanism is supposed to be rigidly connected together, a steady force applied to set the central line of electric particles in motion will not be able to produce in them the full velocity until time has elapsed sufficient to allow the inertia of the connected mechanism to be overcome. We are thus able mechanically to imitate the phenomena of *self-induction* of the circuit and the gradual rise of current strength in an inductive circuit under the operation of a steady impressed electromotive force, and to deduce it as a consequence of the fundamental hypothesis.

Our theory, then, points out that a current should rise gradually in strength, and also that the embracing lines of magnetic force must be considered to come into existence successively as the rotation is taken up in ever-widening circles by the molecular vortices successively receiving motion of rotation. Also, on withdrawing the impressed electromotive force the inertia of the mechanism tends to make it run on for a little, and the electric particles, which by their motion started the vortices, are now themselves urged forward for a little in the same direction, and this constitutes the *extra current* at "break." Let us next endeavour to see what ought to happen on the supposition that there are two conducting circuits in the field, both forming closed circuits, and to one of which an impressed electromotive can be applied. Let V_1, V_2, V_3, &c. (Fig. 3), represent the sectional view of a series of vortex lines of electric cells, and let I_1, I_2, I_3, &c., be the idle wheels or electric particles. Let the row of electric particles I_1 be supposed to be lying inside a conducting circuit, A, represented by the dotted lines, and by our fundamental supposition, the particles I_1 are quite free to move along the conductor, and to rotate on their axes.

Let there be another conductor, B, placed parallel to A, and let I_5 be the electric particles in it. The space C between is supposed to be occupied by a dielectric, and in it the electric particles can only be displaced elastically from a fixed position. We may regard these idle wheels I_2 I_3 I_4 as tethered by springs to one spot. Such being the mechanism, imagine that the row of particles I_1 is urged forward in a downward direction. As the row of particles pass between the cells V_1 V_2 they will set them in rotation in opposite directions. Owing to the inertia of the vortices the first effect of the rotation of V_2 will be to cause I_2 to roll over V_3 and be displaced in an upward direction; its displacement is resisted by the elastic force of the spring. The rotation of I_2, however, sets V_3 in rotation,

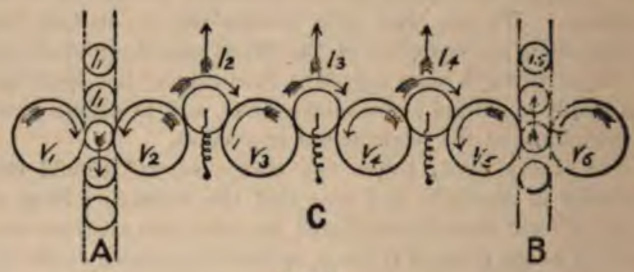


FIG. 3.

and after a short interval V_3 is rotating at the same speed and in the same direction as V_2. I_2 then ceases to be displaced, because the action of V_2 on I_2, and the reaction of V_3 on I_2, simply amount to a *couple* or *twist* on I_2. The same sort of action results in a gradual handing on of the rotation from vortex to vortex, and a propagation of displacement from one idle wheel to the other. When the motion reaches the conductor B, the first result is to cause a displacement of the electric particles upwards, the rotation of V_5 not being instantly acquired by V_6. This amounts to a current in the upward or opposite direction. As soon, however, as the vortex V_6 has accepted the full speed of rotation, then the forces on the electric particles I_5 amount only to *twists*, and not to forces of displacement; hence, the particles I_5 cease to experience any force impelling them forward, and come to rest in virtue of the fact that the conductor offers a resistance to their motion. They fritter down their energy of motion into heat, and come to rest. Hence, the induction current in the conductor after a short flow ceases, and the vortex spin becomes equal in the vortices on either side of it. Suppose now that the impressed force in the circuit A is withdrawn, the electric particles in the A circuit are driven forward for a short time by the energy stored up in the adjacent vortices; these last, however, give up one by one their energy to the circuit A, where it is dissipated as heat. This surrender of velocity is propagated outwards until at the surface of the circuit B the state of things finally is, that when the vortex V_5 has come nearly to rest, the motion of V_6 still continues. The energy of V_6 and of vortices beyond expends itself in moving forward the electric particles in circuit B, in the same direction as that in which the current in A was travelling originally—in other words, part of the energy of the field is spent in making a transitory current in B as well as in A in the same direction. It follows, therefore, that there is a less induction current in A at breaking circuit when a closed circuit B is present than if B were not there—that is to say, the presence of a closed secondary circuit, B, diminishes the self-induction of the primary circuit, as is known to be the case. We see, therefore, that the theory is so far in accordance with observed facts.

The theory must, however, be taken for no more than it is worth, viz., an attempt to construct a mechanical system which shall act in the manner in which we find electro-magnetic fields and circuits do act. The true mechanism *may* be very different; the one described has at least the utility that it shows a way in which the observed effects might be produced. The various dynamical elements in the supposed mechanism have their equivalents in the recognised electrical and electro-magnetic qualities. The angular velocity of the cells or vortices around their axes represents the intensity of the magnetic force, or the strength of the magnetic field. The angular momentum of the vortices represents the magnetic induction, and hence the mass of each cell, or the *density* of the medium, is the analogue of the magnetic permeability. This is greater in paramagnetic substances than in air or vacuum, and greatest of all in iron; in fact, so exceptional is it in iron that Maxwell supposed the particles of the iron themselves to take part in the vortex action. Hence, the energy of a magnetic field is greater if that field contain iron, and accordingly the presence of iron in a core immensely increases the vortex energy for a given vortex velocity, that is, it increases the inductance of the circuit. The energy associated with any revolving cell or vortex is proportional to the product of its velocity and momentum, or the product of the magnetic force, and the magnetic induction estimated in the same direction is a measure of the energy per unit of volume existing in that portion of the field. The "number of lines of force" passing through any circuit is on this theory to be identified with the whole momentum of the molecular vortices linked with that circuit. If any circuit is traversed by lines of force or linked with lines of molecular vortices, and the cause creating this field is removed, say, by withdrawing the magnet or repressing the electric current creating it, the vortices give up their energy gradually to this secondary circuit, and it appears there as energy of motion of the electric particles or as electric current. When one system of bodies in motion sets another set in motion by mutual action and reaction, and there is no loss of energy by anything like friction or imperfect elasticity, then the momentum gained by one must be equal to that lost by the other, and the rate of gain of momentum of the one system is at any instant equal to the rate of loss of momentum by the other. Hence, if the vortices lose momentum their rate of loss of momentum—that is, the rate of withdrawal of lines of induction from the circuit, must be equal to the rate of gain of momentum of, or to the force acting on, the electric particles which are absorbing the momentum. Hence we see that the impressed electromotive force in the circuit must be equal to the rate of withdrawal of lines of induction, and the theory conducts us to Faraday's law of induction as a necessary dynamical consequence of our fundamental assumption. Maxwell has extended the theory of molecular vortices to the explanation of electrostatic phenomena, with which we are not, however, here directly concerned. We have seen that the theory is capable of affording an explanation on mechanical principles of self-induction, mutual induction, and the law of electro-magnetic induction. In order to complete the theory as far as regards the phenomena of magnetism, it is necessary to suppose that the particles of magnetisable metals, such as iron, are set in rotation by the molecular vortices which traverse them, and that an increase of speed of these vortices does not increase proportionally the rotation of the iron molecules. These last behave like wheels slung loosely on a shaft, between which shaft and the wheel there is friction decreasing as the speed of rotation of the shaft increases. If, then, the wheel experiences a constant frictional resistance from external causes, indefinite increase of speed of the shaft would accelerate the wheel's rotational velocity up to a certain point, and the wheel would then cease to rotate. This supposition would enable us to make our theory agree with the fact that increase of magnetic force does not increase indefinitely the magnetic induction through iron, but brings it up to a point at which, approximately speaking, the induction remains stationary. To sum up, we may say that the hypothesis of molecular vortices is an endeavour to imagine a mechanism capable of accounting for electro-magnetic induction on dynamical principles, and on the assumption that the energy of a magnetic field is energy stored up in a medium in virtue of a particular kind of rotation of its parts.

This medium consists of portions capable of elastic displacement when we consider parts of it lying in dielectrics or capable of progressive movement when in conductors, and these portions constitute what we call electricity. Other portions are capable of rotation round closed axes of rotation, and these constitute what we call "lines of force." The medium possesses, therefore, an elastic resilience, and the reciprocal of this quality, or its freedom of yielding to electromotive force, is recognised as the *specific inductive capacity*. The medium possesses also density, and we call this its magnetic permeability, or magnetic inductance. The mass of unit of length of the vortices is equal for all vortices, whether in vacuum, air, or non-magnetic bodies, but in iron the vortices are *loaded* by the adhesion to them of the molecules of the metal, and the density is increased, and hence the permeability; but for very great angular velocities—that is, for great magnetic forces—the adhesion of the molecules and vortices must be supposed to cease, and the permeability approximates to unity. The magnetic force at any point in a field is the angular velocity of the vortex motion at that point, and the magnetic induction is the angular momentum. Magnetic attraction and repulsion is due to the tension set up along a vortex line by the polar contraction and equatorial expansion of the vortex cells. At places where there is magnetic polarity or free magnetism there is a discontinuity in the angular velocity of the vortices within and without the iron. Self-induction is the result of the *inertia* of the molecular vortices, whereby motion set up in them cannot be generated or checked instantaneously. Mutual induction, or the production of induction currents, is due to the fact that differences in the angular velocity of adjacent vortex filaments or cells causes a displacement of the electric particles or idle wheels. Finally, electromotive force is the force causing displacement of the electric particles, and electric currents consist in continuous or periodic movements of these electric particles. Electric currents always produce magnetic fields, because there is nothing of the nature of *slip* between the particles and cells, and, therefore, any progressive movement of the first sets up rotation in the second, and conversely differential rotations or spins of the cells or vortices sets up displacement of the electric particles, causing either electric strain in a dielectric or electric current in a conductor.

§ 4. **Comparison of Theory and Experiment.**—The test of any physical theory is its power to predict new phenomena, or experimental results. The theory of molecular vortices leads to the conclusion that electro-magnetic induction must be propagated through the medium with a finite velocity, and that in dielectrics of unit permeability the velocity of propagation is inversely as the square root of the specific inductive capacity. In the dynamical theory of light it is shown that the ratio of the velocity of light in vacuo to its velocity in any given transparent medium is a constant quantity for each definite wave length, and is called the *index of refraction* of that body for that wave length, and is denoted in physical optics by the symbol μ. Hence, the velocity of light of definite wave length is inversely as the refractive index for that wave length. The refractive index for very long wave lengths can be calculated from observed values of μ for definite rays, and hence numbers obtained representing the relative velocity of these undulations in various transparent bodies. The values of the *dielectric constants*, or reciprocal of the electric elasticities, of various transparent and semi-transparent bodies have also been determined, and it has been found that for a large group of bodies there is a tolerably close agreement between the values of the square root of the dielectric constant and the index of refraction μ_{∞} for very long waves, as shown by the selection from the results of some experimental determinations given in Table A.

TABLE A.

K (Dielectric Constant).	$\sqrt{K}$	μ_{∞} (Refractive Index).	Authority.	Reference.
Sulphur ....3·84	1·96	2·04	Boltzmann	*Pogg. Ann.* CLI., 1874, p. 482.
Colophonium......2·55	1·59	1·54		
Paraffin ....2·32	1·52	1·54		
Pure Rubber......2·12	1·45	1·50	Schiller	*Pogg. Ann.* CLII., p. 535.
Oil of Turpentine 2·21	1·49	1·46	Silow	*Pogg. Ann.* CLVI., 1875, p. 395.
Petroleum ....2·037	1·43	1·46		
Benzene ....2·198	1·48	1·48		
Petroleum Spirit 1·92	1·38	1·38	J. Hopkinson	*Trans.* Roy. Soc. 1877, 1878 and 1881.
Petroleum Oil ...2·07	1·44	1·44		
Ozokerite ....2·13	1·46	1·44		
Turpentine ......2·23	1·49	1·46		

For some other dielectrics, such as glass and the vegetable and animal oils, the agreement is not by any means so close, but for gases, as determined by Boltzmann (*Pogg. Ann.* CLI., 1875, p. 403), there is a fair coincidence. (See Table B.)

TABLE B.

Gas.	K	$\sqrt{K}$	μ
Air....................................	1·00059	1·000295	1·000294
Carbonic acid	1·000946	1·000473	1·000449
Hydrogen	1·000264	1·000132	1·000138
Carbonic oxide.....................	1·000690	1·000345	1·000340
Nitrous oxide	1·000994	1·000497	1·000503
Olefiant gas	1·001312	1·000656	1·000678
Marsh gas	1·000944	1·000472	1·000443

The gases are taken at 0°C. and 760 millimetres pressure. Accordingly, we can say that, for a large group of dielectrics, of which the magnetic permeability is unity, and hence the velocity of propagation of an electro-magnetic impulse proportional to the square root of the electric elasticity or to the reciprocal of the square root of the dielectric constant, we do find a fair agreement between these numbers and the numbers representing the refractive indices or the relative velocities of propagation of very long waves or disturbances in the ethereal medium postulated to account for the phenomena of light. The imperfect agreement between the values of refractive index for long wave lengths and the square root of the dielectric constant for some other bodies shows that the theory is only approximately in agreement with fact, and that the results obtained by the methods adopted for determining the dielectric constant are perhaps impure, and do not give the true value of the electric elasticity. When we consider that the displacements which constitute the light wave motion of the luminiferous ether are changed some billions of times per second, it is seen to be highly probable that measurements of the specific inductive capacity in which the electric stresses are only reversed tens or hundreds of times in a second may be rendered impure or mixed owing to the presence of effects due to an imperfect electric elasticity introduced by the superposition of electric conduction or of electrolytic transport upon the true or elastic displacement effect. In fact those bodies, such as glass and the vegetable oils, which exhibit the greatest discrepancy are those in which the chemical composition indicates a possibility of electrolysis. There may be an electric displacement in such electrolysable bodies over and above the true electrostatic displacement which is engendered by a molecular change in the body, which change results in actual decomposition when the electric force reaches a certain limit. Put broadly, it may amount to this, that the true electric displacement is a displacement of electricity within the molecule, but that in electrolysable bodies electric stress sets up a strain of the molecule itself which, within certain limits, is an elastic strain, and disappears with the removal of the stress, but that beyond these limits molecular disruption takes place. In these cases the displacement measured in taking the specific inductive capacity is the true or dielectric displacement *plus* a displacement due to strain of the molecule, and the result would be to make K appear too great, and, in fact, for glass and certain oils the values in Table C have been obtained, which in all cases are such that $\sqrt{K}$ exceeds the value of μ_∞, or the refractive index, for very long waves of light.*

TABLE C.

Substance.	K	$\sqrt{K}$	μ_∞ (approx.)
Glass, extra dense flint	9·896	3·1	1·5 to 1·6
" light flint	6·72	2·59	
" crown	6·96	2·63	
" plate	8·45	2·90	
Castor oil	4·78	2·18	1·46
Sperm "	3·02	1·73	1·46
Olive "	3·16	1·77	1·46
Neatsfoot oil	3·07	1·75	1·45

J. Klemencic (abstract in the *Journal of the Society of Telegraph Engineers*, 1886, p. 108) has experimented also on the specific inductive capacity of gases and vapours, and given a table in which he compares $\sqrt{K}$ with μ (refractive index) of these same bodies. It is seen that the agreement of $\sqrt{K}$ and μ is very close for the simple gases, but that a marked difference exists in the case of more complicated molecules.

Gas.	$\sqrt{K}$ Boltzman.	$\sqrt{K}$ Klemencic.	μ Refractive index.
Air	1·000295	1·000293	1·000293
Hydrogen	1·000132	1·000132	1·000139
Carbonic acid	1·000473	1·000492	1·000454
Carbonic oxide	1·000345	1·000347	1·000335
Nitrous oxide	1·000497	1·000579	1·000516
Olefiant gas	1·000656	1·000729	1·000720
Marsh gas	1·000472	1·000476	1·000442
Carbonic bisulphide	—	1·001450	1·001478
Sulphurous acid	—	1·00477	1·000703
Ether	—	1·00372	1·00154
Ethyl chloride	—	1·00776	1·001174
Ethyl bromide	—	1·00773	1·00122

The specific inductive capacity of a vacuum is taken as unity, and Boltzman's values are given for comparison.

* *See* Dr. J. Hopkinson, *Phil. Trans. Roy. Soc.*, Vol. CLXXII., 1881, p. 372.

§ 5. Velocity of Propagation of an Electro-Magnetic Disturbance.—There is another line of experimental inquiry which leads to an important relation between electric and optic phenomena. This is the comparison of electrostatic and electromagnetic measurements. If two very small spheres are electrostatically charged and placed with their centres at a unit of distance apart, the stress between them may be mechanically measured. If the conductors are equally charged with opposite kinds of electricity, and the stress when at a unit of distance in air is one unit, the electric quantities are said to be unit electrostatic quantities. If such unit quantities are discharged through a conductor at the rate of one discharge per second, the resulting flow or current is called an electrostatic unit of current.

In the above definition we suppose the dielectric to be a vacuum or some substance such as air, of which the dielectric constant does not differ sensibly from unity. If q and q^1 be two quantities measured electrostatically, and then be placed on small conductors separated by a distance r in a dielectric of constant K, the dynamical force between them will be numerically equal to $\frac{q\,q^1}{K\,r^2}$; and if $q=q^1$, then the force is $\frac{q^2}{K\,r^2}$. Hence, if r is always taken equal to unity, the real quantity of electricity producing by its action on another equal quantity a unit of force will vary as the square root of K when the experiment is performed in various dielectrics. In other words, the absolute magnitude of the electrostatic unit of quantity, and therefore also of current, will vary as the square root of the specific inductive capacity of the medium in which the charges exist. There is another mode in which a unit of current may be defined, and this depends on the definition of a *unit magnetic pole*. If two magnetic poles of equal strength, m, are placed at a distance, r, apart in a magnetic medium of permeability μ, the stress or force between them will be numerically equal to $\frac{m^2}{\mu\,r^2}$, in which expression it is seen that m and μ appear as quantities analogous to q and K in the electrostatic analogue. Hence, when r is unity, we see that to produce a unit stress between the poles m the pole strength must vary as the square root of μ, or the absolute magnitude of the unit magnetic pole varies directly as the square root of the magnetic inductive capacity of the medium in which the experiment is performed, the absolute unit magnetic pole being defined as a pole which at a unit of distance acts on another like pole with a unit of force in a magnetic medium, assumed to be vacuum, or some standard substance of unit permeability.

Since an electric current produces a magnetic force, it may be defined as to magnitude by agreeing that the unit of current is to be one which, when flowing in a circular circuit of unit radius, acts for every unit of length of that circuit with a unit of force on a unit magnetic pole placed at the centre of that circle. The magnitude of the force on the magnetic pole is proportional to the product of the strength of the pole and the strength of the current. Hence, if the magnitude of the unit pole is varied the magnitude of the unit of current will vary inversely as the magnitude of the strength of magnetic pole which is taken as the unit pole. When the medium is varied, the magnitude of the unit magnetic pole, or of the pole which fulfils the condition of acting on another equal pole at a unit of distance with a unit of force, varies directly as the square root of the permeability of the medium. It follows, then, that the magnitude of the *electro-magnetic unit of current* varies inversely as the square root of the magnetic permeability of the medium in which the experiment is made.

We have, then, that the electrostatic unit of current is a quantity which varies directly as the square root of the electrostatic inductive capacity of the medium, or as $\sqrt{K}$, and the electro-magnetic unit of current is another unit of current which varies inversely as the square root of the magnetic inductive capacity of the medium, or as $\sqrt{\mu}$. The electrostatic unit of current represents a much smaller quantity of electricity per second than the electro-magnetic—in other words, the value of the ratio of the magnitude of the unit electro-magnetic current, based on the definition of a unit magnetic pole, to the magnitude of the unit electrostatic current, based on the definition of a unit of electrostatic quantity, is an integer number, and a large one. This ratio of the two units of current varies when the fundamental inductive capacities of the medium is changed, but so that the ratio of the electro-magnetic to electrostatic unit varies inversely as the square root of the product of K and μ. If C_m is the magnitude of the electro-magnetic unit of current, and C_s is that of the electrostatic unit for the standard dielectric, in which $K=1$ and $\mu=1$, then, when the dielectric is changed, $\frac{C_m}{C_s}$ is changed in the ratio of $1 : \sqrt{K\mu}$. Let R_{vac} denote the value of the ratio for vacuum or for a standard dielectric, of which $K=1$ and $\mu=1$, and R_m denote its value for any other medium of which the dielectric constant is K and the magnetic constant μ, then

$$R_m = \frac{R_{vac}}{\sqrt{K\mu}}.$$

We have next to consider what is the physical meaning of this *ratio* of the electro-magnetic and electrostatic units.

The degree in which one quantity is greater or less than another, or to put it more precisely, that amount of stretching or squeezing which must be applied to the latter in order to produce the former, is called the *ratio* of the two quantities.* The ratio of two physical quantities is therefore the expression of the operation which must be performed on the one to make it the physical equivalent of the other. What operation must be performed on an electrostatically measured unit of electricity to make it the equivalent in every way of an electromagnetically measured unit of electricity? The reply is, it must be set in motion with a definite velocity. The electric current produces a magnetic field. The electro-magnetic measure of current is obtained by defining the field by stating its dynamical effect on a defined magnetic pole, and the unit of electric quantity measured electro-magnetically is the quantity conveyed by the unit current so measured in a unit of time. If we imagine a circular or other conductor conveying a unit (electro-magnetic) current to have stretched alongside of it another closely adjacent conductor of like form, each unit of length of which is charged electrostatically with a unit (electrostatic) of electric quantity, we might submit the following question:—The current flowing in the first-named conductor transmits a unit (electro-magnetic) quantity of electricity across each section of it per unit of time: with what *velocity* must the

* W. K. Clifford, *The Common Sense of the Exact Sciences*, p. 99.

electricity in the second conductor be set flowing in order that there may be an equality in the quantities flowing past any sections in each of the conductors, as evidenced by equality in the magnetic fields produced by the first-named current and the moving electric charge? This *velocity* is evidently a concrete velocity, which depends on the very nature of the qualities of the medium which determine magnetic and electrostatic attraction, and this velocity may be called the ratio of the magnitude of the electro-magnetic to the electrostatic unit of quantity. This velocity is evidently one which is determined by the nature of the medium, and not by the particular units of length, time, and mass selected for use in the measurements. This comparison assumes that a moving electrostatic charge is in effect the equivalent of an electric current. This has been put to the test of experiment by Prof. Rowland.* A rigid gilt ebonite disc was fixed to an axis, and could be rotated between two gilt glass discs. One member of a very delicate astatic system of magnetic needles was placed near the disc and shielded from electrostatic disturbance. On charging the gilt ebonite disc and setting it in rapid rotation it was found to affect the magnetic needle whilst rotating just as a current of electricity would have done if flowing in a circular conductor coinciding in form with the periphery of the disc. Since 1876 Prof. Rowland has again in the United States repeated the experiment and confirmed the general result. There is, therefore, experimental foundation for the view that a static charge of electricity conveyed on a moving body creates a magnetic field *whilst it is in movement*. This kind of electric current, in which a static charge is bodily moved on a conductor, is called a *convection current*. The experiment of comparing the magnitudes of an electrostatic and an electro-magnetic unit of electric quantity as above defined was first made by Profs. Weber and Kohlrausch, and the value of that ratio for a medium such as air, in which approximately we have K and μ both equal to unity, gave as a result a velocity very nearly identical with the velocity of light. Since that time very many experimentalists

* See *Phil. Mag.*, 1876, Vol. II., Fifth series, p. 233: Dr. Helmholtz, *On the Electro-Magnetic Action of Electric Convection*. These experiments of Prof. Rowland were carried out at Berlin.

have determined the value of this ratio, which is denoted by the symbol "v." Altogether about a dozen observers have determined it, with the results set out in the Table on the next page.

The best determination of the velocity of light is that made by Prof. Newcomb, at Washington, in 1882. The method employed was the revolving mirror method of Foucault, the distance between the revolving and fixed mirror being in one portion of the experiments 2,550 metres, and in the other portion 3,720 metres. The resulting velocity of light *in vacuo* is $2·99860 \times 10^{10}$ centimetres per second.

The following other results are abstracted from Prof. Everett's book, "Units and Physical Constants," 2nd edition :—

Observer.	Velocity in centimetres per second.
Michelson, at Naval Academy, 1879	$2·99910 \times 10^{10}$
Michelson, at Cleveland, 1882	$2·99853 \times 10^{10}$
Newcomb, at Washington, 1882 (best results)	$2·99860 \times 10^{10}$
Newcomb (other results)	$2·99810 \times 10^{10}$
Foucault, at Paris, 1862	$2·98000 \times 10^{10}$
Cornu, at Paris, 1874	$2·98500 \times 10^{10}$
Cornu, at Paris, 1878	$3·004 \times 10^{10}$
Last result discussed by Listing	$2·9999 \times 10^{10}$
Young and Forbes, 1880-81	$3·01382 \times 10^{10}$

Earlier observations gave as follows :—

Roemer's method, by Jupiter's satellites	$3·000 \times 10^{10}$
Bradley's method, by stellar aberration	$2·977 \times 10^{10}$
Fizeau	$3·142 \times 10^{10}$

The general result of the best determinations is that the velocity of light is very close to $3·000 \times 10^{10}$ centimetres per second, or nearly one thousand million *feet* per second.

We have, therefore, the following facts :—The velocity of light of definite wave length in any medium V_m is connected with the velocity V_v of the same ray in vacuo by an equation—

$$V_m = \frac{V_v}{\mu},$$

where μ is the refractive index of that medium for the particular wave length considered, and also that the velocity V_v is very nearly 3×10^{10} centimetres per second. Also we find that the ratio of the electro-magnetic to the electrostatic unit of

NAME.	Reference.	Observed value of "v."	Corrected value (*).
WEBER and KOHLRAUSCH } 1856	*Electrodynamische Maasbestimmungen* and *Pogg. Ann.*, XCIX., August 10, 1856.	$3{\cdot}1074 \times 10^{10}$ Comparison of electric quantities.	$3{\cdot}1074 \times 10^{10}$
SIR W. THOMSON and W. F. KING } 1867 1868	*Report of British Assoc.*, 1869, p. 434; and *Reports on Electrical Standards*, F. Jenkin, p. 186.	$2{\cdot}846 \times 10^{10}$ Comparison of electromotive forces.	$2{\cdot}81 \times 10^{10}$
SIR W. THOMSON and DUGALD MCKICHAN } 1872	*Phil. Trans. Royal Soc.*, 1873, p. 409.	$2{\cdot}93 \times 10^{10}$ Comparison of electromotive forces.	$2{\cdot}89 \times 10^{10}$
CLERK MAXWELL, 1868	*Phil. Trans. Royal Soc.*, 1868, p. 643.	$2{\cdot}88 \times 10^{10}$ Comparison of electromotive forces.	$2{\cdot}84 \times 10^{10}$
AYRTON and PERRY, 1878	*Journal of the Society of Telegraph Engineers*, Vol. VIII., p. 126.	$2{\cdot}98 \times 10^{10}$ Comparison of electric capacities.	$2{\cdot}94 \times 10^{10}$
SIR W. THOMSON and SHIDA } 1880	*Phil. Mag.*, Vol. X., p. 431, 1880.	$2{\cdot}995 \times 10^{10}$ Comparison of electromotive forces.	$2{\cdot}955 \times 10^{10}$
J. J. THOMSON, 1883	*Phil. Trans. Royal Soc.*, 1883, p. 707.	$2{\cdot}963 \times 10^{10}$ Comparison of electric capacities.	$2{\cdot}963 \times 10^{10}$
HIMSTEDT, 1888	*Electrician*, March 23, 1888, Vol. XX., p. 530.	$3{\cdot}0074 \times 10^{10}$ Comparison of electric capacities.	$3{\cdot}0074 \times 10^{10}$
KLEMENCIC, 1887	*Proceedings of the Society of Telegraph Engineers*, 1887, p. 162.	$3{\cdot}015 \times 10^{10}$ Comparison of electric capacities.	$3{\cdot}015 \times 10^{10}$
SIR W. THOMSON, AYRTON and PERRY } 1888	*British Association, Bath*, and *Electrician*, September 28, 1888.	$2{\cdot}920 \times 10^{10}$ Comparison of electromotive forces.	$2{\cdot}920 \times 10^{10}$
FISON, 1888	*Electrician*, Vol. XXI., p. 215, and *Proc. Phys. Soc.*, Lond., June 9, 1888.	$2{\cdot}965 \times 10^{10}$ Comparison of electric capacities.	$2{\cdot}965 \times 10^{10}$
SIR W. THOMSON, 1889	*Royal Institution Lecture*, February 8, 1889.	$3{\cdot}004 \times 10^{10}$ Comparison of electromotive forces.	$3{\cdot}004 \times 10^{10}$

* Corrected value of "v" in centimetres per second for value of B.A. unit in terms of the true ohm—1 B.A.U. = $\cdot 9868 \times 10^{9}$ centimetres per second.

electric quantity or current in any dielectric and magnetic medium R_m is connected with the same ratio measured in vacuo R_v by an equation—

$$R_m = \frac{R_v}{\sqrt{K\mu}},$$

where K is the dielectric constant and μ the magnetic permeability.* Experiment has also indicated that within narrow limits, taking best results, R_v and V_v have the same value, namely, 3×10^{10} centimetres per second, and that $\sqrt{K}$ has the same value as μ (refractive index) for media, for which μ (permeability) has the value unity. We are led, therefore, to infer that this close relationship is not a matter of accident, but that it indicates a very intimate connection between electricity and light, and that the hypothesis that light is a disturbance propagated through an elastic medium may be supplemented with some considerable show of reason by the hypothesis that electro-magnetic phenomena are the result of actions taking place in identically the *same medium* or ether. There are no transparent media for which the magnetic permeability differs by more than a very small quantity from unity, and hence the approximate identity of the values of the ratio of the units compared in air with the value of the velocity of light waves of very long wave length; and the approximate identity for true dielectrics of the value of the refractive index and of the square root of the dielectric constant furnishes a test of the probability of the truth of the electro-magnetic theory of light. Maxwell's mathematical method of arriving at this theory consisted in forming certain equations expressing the velocity of propagation of *vector potential*, and noticing that these equations were mathematically of the same form as those which determine the velocity of propagation of a disturbance through an elastic medium. The physical meaning of this term, vector potential, may be arrived at as follows :—

Suppose a regiment of soldiers to set off marching down a street, the ranks being well spaced out. At any place in the

* It is unfortunate that usage has consecrated the same Greek letter μ for refractivity in optics and magnetic inductivity in electro-magnetics. In some respects it would be an advantage in electro-optics if these quantities were differently symbolised.

street let two lines be drawn across the street parallel to each other and a few yards apart. Let two observers take note of how many soldiers cross each line. At any instant the total number of soldiers which are contained between the two lines is equal to the difference between the numbers which have crossed each line respectively. However irregular the movement may be, the total number of soldiers at any instant in the area or the product of the area, and the number of soldiers per unit of area within the boundary, will be equal to the number obtained by reckoning the algebraic sum of the soldiers which have from the beginning of the time crossed the whole boundary line, calling those numbers *positive* when soldiers have stepped *into* the area and *negative* when they have stepped

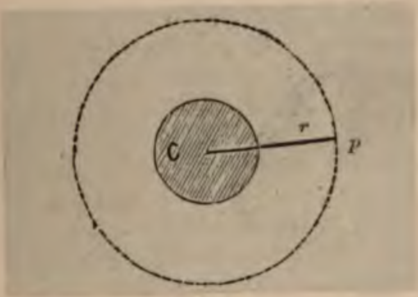


FIG. 4.

out of it. We have here a simple example of the way in which a *line integral* may be the equivalent of a *surface integral*. If the area be irregular in shape and contain A square yards, and if the perimeter be l linear yards, then if n_1 n_2, &c., are the number of men which have stepped across each yard length of the boundary, and if N_1 N_2, &c., are the number of men in respective square yards within the area at any instant, then $N_1 + N_2 +$ &c., to A terms or ΣN is called a surface integral and will be equal to $n_1 + n_2 +$ &c., to l terms, which is a line integral, provided that each n is reckoned positive when men step in, and negative when men step out of the area over each yard of the boundary. The algebraic sum of all the stepping over the boundary all the way *round* the area is equal to the sum of the men per square yard all *over* the area. We have here given an illustration of an important proposition on mathematical physics, viz., that a surface integral, or the summation of a certain quantity *over* an area, can be replaced by a line integral, or the summation of another related quantity all *along* the boundary line of that area. We proceed to illustrate it from an electrical point of view.

Let C (Fig. 4) be the circular cross-section of an infinite straight wire conveying a current C. Round C describe a circle of radius r. The magnetic force at p is known to be equal to $\frac{2\,C}{r}$ units, and is directed along the circumference of the circle; the line integral of the magnetic force along the

Fig. 5.

dotted line is equal to $\frac{2\,C}{r} \times 2\,\pi\,r = 4\,\pi\,C$, and the surface integral of the current through the area enclosed by the dotted circle is C. Hence we have generally that the line integral of the magnetic force is equal to $4\,\pi$ times the surface integral of the current. This proposition is generally true, and it is easy to show that if A be any area (*see* Fig. 5) traversed normally by a current, such that the current density is u over any element of area $d\,s$, then the integral of $u\,d\,s$ all over the area, or $\int u\,d\,s$, is equal to the line integral of the magnetic force taken along the boundary line. The mathematical operation of taking a line integral has been called by Maxwell *curling*, and we express the above proposition by saying that $4\,\pi$ times the total current through the area is equal to the *curl* of the magnetic force round it. On the theory that lines of magnetic force do not spring suddenly into existence in a field, but are propagated onwards from point to point in the field, it is possible to show that just as the current is the curl of the magnetic force so the magnetic force is the curl of another quantity called the *vector potential.*

Let A B (Fig. 6) be a portion of a straight conductor in which a current can be started. Let $x\,x'$, $y\,y'$ be two lines drawn a unit of distance apart, parallel to each other and at right angles to the conductor. These lines bound a strip of plane space taken in the plane of the current. Draw any two transverse lines $a\,b$, $c\,d$, parallel to the conductor and separated by a small

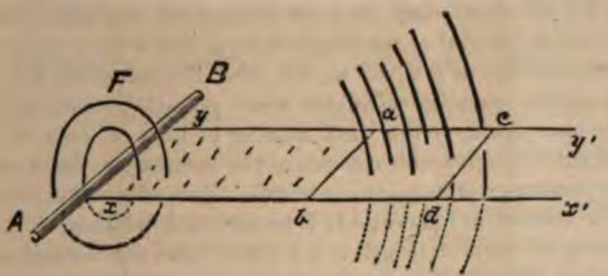


FIG. 6.

distance. We know that when a current is started in the conductor the lines of magnetic force F will be circles formed round A B as axis, and having their planes perpendicular to the plane $x\,x'$, $y\,y'$. Let us now assume that if a current is suddenly started in the conductor A B the magnetic force is propagated outwards from the conductor with a finite velocity v. In other words, each circular line of force must be considered to expand outwards like a circular ripple on the surface of water. When once the field has arrived everywhere at its normal value the magnetic force at a distance r from the wire is $\frac{2\,C}{r}$, where C is the value of the current, and we shall suppose, as usual, that the magnetic field is indicated as to value by the density of the lines of force, or that the number per

AA 2

square centimetre traversing normally the plane $x\,x', y\,y'$ is at any point proportional or numerically equal to the magnetic force at that point. If, then, we neglect for the moment all effect of self-induction, and suppose the current *in* the wire to rise up instantaneously to its full value, we may yet regard the circular lines of force as expanding outwards with a certain velocity of enlargement, and attaining or taking up their final positions after a short interval of time. If we represent the intersections of these rings of force on the plane of $x\,x'\,y\,y'$ by dots, these dots will march forward like the soldiers in the previous illustration. The total number of lines of force which at any instant are found traversing the area $a\,b\,d\,c$ is equal numerically to the difference in the number between those which from the beginning of the epoch have intersected or cut through the line $a\,b$ and those which have cut through $c\,d$. In other words, the surface integral of the magnetic force over $a\,b\,c\,d$ may be represented by, or is equal to, the line integral round $a\,b\,c\,d$ of a certain quantity called the *vector potential*, which, physically interpreted, is the total number of lines of force which have cut through a unit element of the boundary in the process of expansion or propagation outwards. This term vector potential is justified as follows:—If F be the total number of lines of force per unit of length of $a\,b$ which have cut through $a\,b$ from the instant of beginning the current, and if the small distance $b\,d$ is called δx, the length $x\,b$ being called x, then by Taylor's theorem (Diff. Calc.), the number which have cut through unit of length of $c\,d$ is $\mathrm{F} - \frac{d\,\mathrm{F}}{d\,x}\,\delta x$, and hence the difference between F and this last quantity is $\frac{d\,\mathrm{F}}{d\,x}\,\delta x$, and this last when multiplied by δy, which we may take for the length of $a\,b$ or $c\,d$—that is $\frac{d\,\mathrm{F}}{d\,x}\,\delta x\,\delta y$—is the total number of lines of force included in the area $a\,b\,c\,d$. If we call the induction through this area B—that is to say, the number of lines of force per square centimetre is B—it follows that the number through $a\,b\,c\,d$ is $\mathrm{B}\,\delta x\,\delta y$. Hence, equating the two values

we have $$\frac{d\,\mathrm{F}}{d\,x}\,\delta x\,\delta y = \mathrm{B}\,\delta x\,\delta y,$$

or $$\frac{d\,\mathrm{F}}{d\,x} = \mathrm{B}.$$

Hence, the mean magnetic force over the small area is numerically equal to the space variation of a certain quantity F. In electrostatics the electric force X at any point in the electric field is the space variation of a certain quantity V, called the electrostatic or scalar potential—that is to say,

$$-\frac{dV}{dx}=X;$$

and accordingly by analogy that quantity F whose space variation gives the magnetic force under the circumstances considered above is called the *vector potential of the current.* From Ampère's investigations it is known that the magnetic force due to an element of a current C of length δs at a distance r from this element has the value $\frac{C\,\delta s}{r^2}$, and is along a line at right angles to the plane containing δs and r. The space variation of $-\frac{C\,\delta s}{r}$ is $\frac{C\,\delta s}{r^2}$; hence the vector potential of an element of current at any point is proportional to the length of that element divided by its distance from that point.

In electrostatic phenomena we obtain the static potential at any point due to any charge Q by taking each element q of the charge, and dividing the magnitude of this element of charge by its distance from the point at which the potential is required, and taking the sum $\Sigma\frac{q}{r}$ of all such quotients. In electrostatics the potential at a point is a *scalar* or directionless quantity, and the summation is merely an algebraic sum; but in dealing with currents the quotients $\frac{C\,\delta s}{r}$ are vectors, or directed quantities, and have to be added together according to the laws for the addition of vector quantities just as forces and velocities are added. Hence the potential of a current at any point is a vector or directed quantity. The lines of vector potential of a straight current are lines described in space parallel to the current, and the lines of vector potential of a circular current are circles described on planes parallel to the plane of the current. Returning to the simple case of a straight current, let us suppose that a unit of length is described somewhere parallel to the current, and that on starting the current suddenly circular lines of magnetic force are propagated outwards with a velocity V; these lines will, as they expand, cut perpendicularly through the element of length just as the expanding ripples on water due to a stone dropped into it would "cut through" a stick held perpendicularly in the water a little way from the place where the "splash" was made. Suppose that after N lines of force have cut through the element of length this little line is made to move forward parallel to itself, so that there is no further increase in the number of lines of force which afterwards cut through it, it is evident that it must move with the velocity of propagation of the expanding rings of force. But the number expressing the number of lines of force which have cut through the element of length already is the value of the vector potential at that point where the element is at that instant; hence the velocity of propagation of the vector potential *is* the velocity of propagation of an electro-magnetic disturbance. Maxwell's general mathematical method of investigating the propagation of an electro-magnetic disturbance consisted in forming equations expressing the change of the value of the vector potential of a current or system of currents at any point in the field, and deducing equations which mathematically are of the same type as those which express the propagation of a disturbance through an elastic solid or fluid, and his result was that the velocity of propagation of the vector potential through a medium of electrostatic and magnetic inductivities K and μ was equal to $\frac{1}{\sqrt{K\mu}}$, or to $(K\mu)^{-\frac{1}{2}}$.

The complete proof of the above proposition as given by Maxwell in all its generality requires some elaborate analysis, but it is not difficult to give a simple illustration by treating a reduced case, and which will exhibit the principles of the more complete problem.

Let an infinite straight conductor be supposed situated in a dielectric medium of specific inductive capacity (electrostatic inductivity) K and of permeability (magnetic inductivity) μ. We proceed to investigate the velocity of lateral propagation of electro-magnetic induction on the supposition that if a current is instantaneously started at its full value in the conductor, supposing this possible, the magnetic force travels outwards laterally from the conductor in all directions with a velocity v. This amounts to the supposition that the circular lines of magnetic force surrounding the conductor swell out or expand outwards from the surface of the conductor, so that the radius of any determinate circular line of force increases or grows with a velocity v. It must be borne in mind that the magnetic force at any point in the field at any instant is defined by the density or concentration of the lines of force—that is, by the number passing normally through a unit of area. If we complicate the problem by supposing the strength of the current in the conductor to gradually increase, then the concentration of the lines at any point must be supposed to increase gradually, but the rate of increase of concentration—that is, of the force—

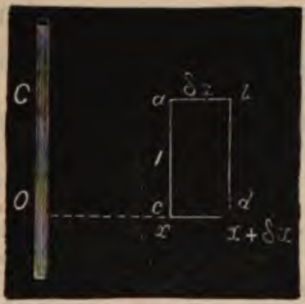


FIG. 7.

is a different thing from the rate of outward movement of the lines of force.

We might in imagination suppose each line of force to be labelled so as to recognise it. All the lines travel outward from the conductor at the same rate, but some go out farther than others. The first ones shed off expand out to reach positions in the most distant portions of the field, and the succeeding ones reach intermediate positions, and as the current strength grows up fresh arrivals or deliveries of lines of force happen which pack the space fuller, and increase the concentration at all points of the field, at a rate depending on the rate of growth of the current.

Let O C (Fig. 7) be a portion of the straight conductor. In the plane of O C take any little rectangular area $a\,b\,c\,d$, with side $a\,c$ equal to unit of length, and side $a\,b$ equal to δx, δx being a very small quantity compared with the distance between O C and $a\,c$—that is, let the distance $O\,c = x$ and $O\,d = x + \delta x$, and let the distance δx be the distance by which the radius of any circular line of force of the conductor O C increases in a small time δt. At any instant the number of lines of force which pass normally through the small area $a\,b\,c\,d$ is equal to the difference between the number which have "cut" across $a\,c$ and those which have cut across $b\,d$ in consequence of our supposition as to the outward growth or expansion of the circular lines of force. Let F be the total number of lines of force due to the current in O C which have from the beginning of the current flow "cut across" $a\,c$, then, by the principles of the Differential Calculus, the number which have cut across $b\,d$ is represented by the quantity $F - \frac{dF}{dx}\delta x$, and the number existing in, or perforating through, the area $a\,b\,c\,d$ is the difference between F and $F - \frac{dF}{dx}\delta x$, or equal to $\frac{dF}{dx}\delta x$. Let **B** stand for the induction through unit of area of the rectangle $a\,b\,c\,d$, or to the number of lines of force per unit of area, then the total number of lines of force through $a\,b\,c\,d$ is represented also by $\mathbf{B}\,\delta x$, since the area of $a\,b\,c\,d$ is δx square units, $a\,c = b\,d$ being unity.

Hence, $$\frac{dF}{dx} = \mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or the induction is represented by the space rate of change of the vector potential of the current at that point in the direction of x. In this case let it be borne in mind that the vector potential signifies the number of lines of force which have from the beginning of the epoch cut through unit length taken parallel to the current. Again, since by supposition each line of force moves outwards parallel to itself through a distance δx in a time δt, $\frac{\delta x}{\delta t}$ is the velocity of propagation v of the electro-magnetic disturbance or of the vector potential. The rate of "cutting across" $a\,c$ at any instant is represented by

$\frac{dF}{dt}$; hence the number of lines of force added to the area in a time δt must be $\frac{dF}{dt}\delta t$, and this must be equal to the accumulation of the lines in $a\,b\,c\,d$ in the same time in the area $a\,b\,c\,d$.

If in a small time interval the rate of cutting across $a\,c$ is $\frac{dF}{dt}$, then the rate at which "cutting" is taking place across a length $b\,d$, removed by a distance δx, is

$$\frac{dF}{dt}+\frac{d}{dx}\left(\frac{dF}{dt}\right)\delta x,$$

and the rate at which accumulation of lines in induction is going on in the area is

$$-\frac{d}{dx}\left(\frac{dF}{dt}\right)\delta x.$$

Hence, since B is the induction per unit of area and the area of $a\,b\,c\,d$ is δx square units, the rate of increase of induction through $a\,b\,c\,d$ is

$$\frac{d}{dt}(B\,\delta x).$$

Accordingly we have

$$\frac{d}{dt}(B\,\delta x)=-\frac{d}{dx}\left(\frac{dF}{dt}\right)\delta x,$$

or since δx is constant,

$$\frac{dB}{dt}=-\frac{d}{dx}\left(\frac{dF}{dt}\right),$$

$$=-\frac{d}{dt}\left(\frac{dF}{dt}\right)\frac{dt}{dx},$$

or,

$$\frac{dB}{dx}\frac{dx}{dt}=-\frac{d^2F}{dt^2}\frac{dt}{dx};$$

but $\frac{dx}{dt}=v=$ velocity of propagation of the impulse. Hence,

$$v^2=\frac{-\frac{d^2F}{dt^2}}{\frac{dB}{dx}} \quad \ldots\ldots \quad (2)$$

or, generally, since $B = \frac{dF}{dx}$, we have

$$\frac{d^2 F}{dt^2} + v^2 \frac{d^2 F}{dx^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

as the equation of motion of the electro-magnetic induction. This equation, which is a reduced case of the general one, is of the same type as that obtained in the theory of sound for the propagation of an impulse along a tube or canal. In the case of sound the symbol F would be the *velocity potential.** In the electro-magnetic problem the F is the *vector potential.* It might perhaps be more expressively called the *induction potential.*

The rate of cutting, or the value of $\frac{dF}{dt}$, also expresses the electromotive force acting along the unit of length ac in the dielectric. On Maxwell's hypothesis this electromotive force *in* the dielectric acting parallel to the current in the conductor produces a displacement in the dielectric, such that if E is the electromotive force we have as above

$$\frac{dF}{dt} = E = \frac{4\pi}{K} D,$$

where D is the displacement through unit of area ; hence,

$$\frac{d^2 F}{dt^2} = \frac{4\pi}{K} \frac{dD}{dt}; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and $\frac{dD}{dt}$ is the rate of displacement or the displacement current flowing through unit of area taken perpendicularly to the current in O C at the point considered. Let this displacement current be denoted by u. We have then that $\frac{d^2 F}{dt^2} = \frac{4\pi u}{K}$, K being the dielectric constant of the medium.

Consider now a small parallelopipedon (Fig. 8) or solid rectangle described in the dielectric, of which the sides are respectively $ac = 1$, $cd = \delta x$, $ce = \delta y$.

The effect of the cutting across of this solid rectangle by expanding lines of induction will be to generate in it a displacement current such that the total displacement current parallel to ac and through $cdfe$ will be $u\,dx\,dy$. By a previous theorem

* *See* Besant's *Hydromechanics*, p. 251 (Third Edition).

the line integral of magnetic force round any line is equal to 4π times the surface integral of the current through the area bounded by that line, and this is true whether the magnetic force be produced by that current, or whether it is a current produced by a certain changing magnetic force. Apply the theorem to the small rectangle bounded by the lines *c e f d*. The surface integral of the current through *c e f d* is $u\,dx\,dy$. The magnetic force along *c e* is $\frac{B}{\mu}$, where B is the induction at *c* and μ is the magnetic permeability of the medium, since by a fundamental theorem the magnetic induction B at any place is equal to μ

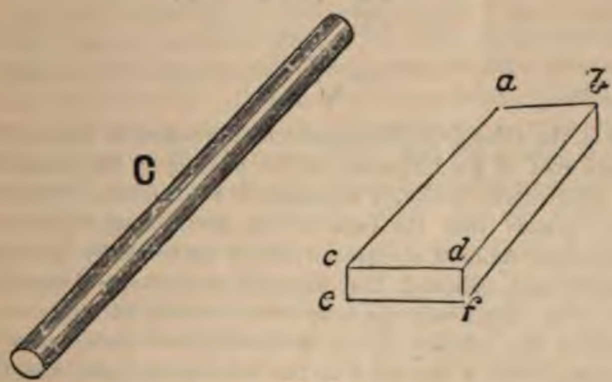


FIG. 8.

times the magnetic force at that point. The magnetic force along *d f* removed by a distance δx from *c e* is $\frac{1}{\mu}\left(B - \frac{dB}{dx}\delta x\right)$, and there is no magnetic force along *c d* and *e f*, for these sides are perpendicular to the direction of the magnetic force of the current in O C. Hence, the line integral of magnetic force round *c e f d* is

$$\frac{1}{\mu}\left(B\,\delta y - (B\,dy - \frac{dB}{dx}\delta x\,\delta y)\right),$$

or

$$\frac{1}{\mu}\frac{dB}{dx}\delta x\,\delta y;$$

hence,

$$4\pi u\,\delta x\,\delta y = \frac{1}{\mu}\frac{dB}{dx}\delta x\,\delta y,$$

or

$$4\pi\mu u = \frac{dB}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Accordingly, in the equations (4) and (5) above, we have obtained values for the quantities $\frac{d^2 F}{dt}$ and $\frac{dB}{dx}$ in terms of the permanent qualities of the medium; and by substitution of these values in equation (2) above we have that the square of the velocity of propagation of the vector potential is

$$v^2 = \frac{\frac{d^2F}{dt^2}}{\frac{dB}{dx}} = \frac{\frac{4\pi u}{K}}{4\pi\mu u} = \frac{1}{K\mu},$$

or

$$v = \sqrt{\frac{1}{K\mu}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

that is, the velocity of propagation of the magnetic force is the square root of the reciprocal of the product of the magnetic and electrostatic inductive constants of the medium. We have above proved that the ratio of the electro-magnetic to the electrostatic units of electric current is expressed by the same quantity, and indicated that accurate experiment shows this ratio to be numerically the same as the velocity of light.

Hence, the velocity of an electro-magnetic disturbance or magnetic force is the same as the velocity of light, and the conclusion is urged upon us with great force that the medium concerned in both phenomena is the same.

§ **6. Electrical Oscillations.**—A survey of the phenomena of electric current induction would be very incomplete if it did not contain some reference to the subject of electrical oscillations. Recent researches has endowed this department of electrical investigation with fresh interest. We proceed to consider the manner in which electrical oscillations may arise. If a material body is subjected to elastic constraint, and is disturbed from a position of equilibrium, it returns when set free to its original position. If that body is endowed with *mass*, and hence possesses the quality of inertia, its motion of return to its position of equilibrium will, under certain circumstances, carry it beyond that point and set up *oscillations*, which decay gradually away. Two illustrations of this readily present themselves, one a mechanical and the other a pneumatical example. The first case is that of a pendulum or straight spring. Let this pendulum or spring be deflected from its position or condition of equilibrium and held in constraint. Next let it be set free—the elastic or restoring forces urge it back again to its first position. In virtue of its mass it will acquire a certain momentum, and on reaching the position of equilibrium this momentum may carry it past this point, and the acquired kinetic energy will then be expended in making a displacement against the elastic forces. If there is nothing of the nature of friction present to fritter away the work expended on the body in making the first displacement, then the energy would remain associated with it for ever, being alternately potential and kinetic, and the oscillations continue with undiminished amplitude. If the spring or pendulum vibrates in a viscous fluid, then a frictional retardation will be experienced, and in so far as this is present the energy is gradually dissipated, and the oscillations decay away, becoming gradually less and less in amplitude. It may so happen that the work done against frictional resistance during the first quarter of a complete oscillation in starting to return from the position of greatest displacement is just equal to the work done in originally making the displacement. When this is the case the whole energy is dissipated by the time the deflected or displaced body reaches its original position of rest, and there are then no oscillations. Accordingly a pendulum or spring may be set in a viscous fluid such that the frictional resistance is just so great that when the body is disturbed and then set free it returns to its original position without ever passing it; in other words, there are no oscillations. Another illustration of oscillatory and non-oscillatory establishment of equilibrium is as follows: Suppose there be two large vessels, or reservoirs, connected by a pipe, closed or closable in the middle by a stopcock. Let one of these vessels, A, be exhausted of its air, and let the other, B, have air in it at the atmospheric, or a greater than the atmospheric pressure. *First*, let the connecting pipe be supposed to be long and narrow; on opening the stopcock air will rush over from B into A, and the flow of air will continue uniformly in the pipe in one direction until the pressure in A and B is equalised. *Second*, let the connecting pipe be very short and large, so that little tubular friction is offered to the flow of air. Under these circumstances the result of opening the top would be that a rush of air would take place, which would be succeeded by a series of oscillations of the air in the tube. The air, in fact, rebounds from side to side, and the equilibrium is only finally established after a series of gradually diminishing oscillations or backward and forward currents of air in the tube. This establishment of equilibrium of pressure by oscillatory movement takes place when the resistance to the flow is small. That this is no fanciful description is proved by the experience of MM. Clément and Désormes in their experiments to determine the ratio of the specific heats of gases. In these experiments a large glass vessel has a partial vacuum made in it. A stopcock is then quickly opened and closed, and the pressure of the air determined after a short time. These experiments were also carried out by MM. Gay Lussac and Welter. See *Journal de Physique*, LXXXIX., 1819, 428, and *Ann. de Ch. et de Phys.* [1], XIX., 1821, 436.

M. Cazin (*Ann. de Ch. et de Phys.* [3] LXVI., 1862, 206) pointed out a source of error which resulted from these air oscillations, and showed that the final pressure depended upon the phase of the oscillation at which the stopcock is closed.

These examples are sufficient to indicate that when a material system of bodies having *inertia* is displaced against elastic forces which compel it to return, if free, to a definite position, whilst at the same time its motion is resisted by actions of the nature of frictional resistance which dissipate its energy, we have a resulting motion which may be oscillatory or non-oscillatory, according to the relation of the constants of the system under certain conditions as to mass, or inertia and friction. We have oscillations dying gradually away. Under other conditions we have a gradual return to the original position without ever passing it. The motion is then said to be perfectly *dead-beat*. We shall investigate presently the conditions which must hold good, and the relations between the *inertia factor*, in virtue of which the moving system possesses kinetic energy, and the *resistance factor*, in virtue of which the energy bestowed upon the system at its first displacement is frittered away into heat, in order that the motion may be vibratory or dead-beat.

When a condenser or Leyden jar is discharged through a conductor, the potential energy runs down in the form of an electric current. In this case we have a similar state of things to that existing when a bent spring is released. This transformation of the potential energy may take place either by a vibratory current or a series of electrical oscillations—that is, by a uni-directional discharge. It is highly probable that Prof. Joseph Henry, as far back as 1842, was the first to recognise that the discharge of a condenser might be of an oscillatory character. It is remarked by him* that "The discharge, whatever may be its nature, is not correctly represented by a single transfer of imponderable fluid from one side of the jar to the other; the phenomena require us to admit the existence of a principal discharge in one direction and then several reflex actions backward and forward, each more feeble than the preceding, until equilibrium is attained. All the facts are shown to be in accordance with this hypothesis, and a ready explanation is afforded by it of a number of phenomena which are to be found in the older works on electricity, but which have until this time remained unexplained." A little later on in the Paper he gives an explanation of the reversal of polarity of the needles by the oscillatory discharge. In his now celebrated Essay, *Erhaltung der Kraft* (Berlin, 1847), Helmholtz alludes also to such a possible form of electric discharge in the following words:—"We assume that the discharge (of a jar) is not a simple motion of the electricity in one direction, but a backward and forward motion between the coatings in oscillation, which become continually smaller until the entire *vis viva* is destroyed by the sum of the resistances." He adds: "The notion that the discharge consists of alternately opposed currents is also favoured by the phenomena observed by Wollaston while attempting to decompose water by electric shocks, that *both* descriptions of gases are evolved at both electrodes." The investigation which, however, marks an epoch in this matter

* "The Scientific Writings of the late Prof. Joseph Henry." Washington: 1886. Vol. I. This statement of Prof. Henry had attention directed to it by Mr. A. D. Raine in *The Electrician* of November 2, 1888, p. 831. It had been previously mentioned, however, in the sketch of the life of Prof. Joseph Henry given in the *Encyclopædia Britannica*, Ninth Edition.

is the Paper by Sir W. Thomson in the June number of the *Philosophical Magazine* for 1853, on "Transient Electric Currents." In this Paper the author discusses, first, the equations which determine these currents at any instant when a condenser or Leyden jar is discharged through a conductor. The discharging conductor is supposed to have self-induction, or, as Sir W. Thomson then called it, "electro-dynamic capacity," and also to have ohmic resistance, which is constant, and independent of the rate of discharge. On these two assumptions he builds up an equation which mathematically contains the whole theory packed up in it, as follows :—

If C is the electrostatic capacity of the jar or condenser, and R the ohmic resistance, and L the constant inductance of the discharging conductor ; and if q is the electric quantity in the jar, and v the potential difference of its coatings at any instant t, then by the definition of electric capacity we have

$$q = C\,v,$$

and $\frac{dq}{dt} = i =$ the current at that instant in the conductor, which is equal by Ohm's law to $\frac{v}{R}$. By the principle of conservation of energy the rate at which electro-magnetic energy is being taken up by the conductor, viz., $\frac{d}{dt}\left(\frac{1}{2}\,L\,i^2\right)$, together with the rate at which energy is being dissipated as heat in the conductor, viz., $R\,i^2$ (by Joules' law), must be equal to the rate of decay of the energy contained in the jar, or to

$$-\frac{d}{dt}\left(\tfrac{1}{2}\,q\,v\right) = -\frac{d}{dt}\left(\tfrac{1}{2}\frac{q^2}{C}\right).$$

Hence
$$-\frac{d}{dt}\left(\tfrac{1}{2}\frac{q^2}{C}\right) = \frac{d}{dt}\left(\tfrac{1}{2}\,L\,i^2\right) + R\,i^2,$$

or
$$-\frac{q}{C}\frac{dq}{dt} = L\,i\,\frac{di}{dt} + R\,i^2;$$

but $i = \frac{dq}{dt}$, or the current is the rate of loss of charge, therefore

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}\,q = 0 \quad . \quad . \quad . \quad \text{(I.)}$$

The value of q, or the charge in the jar at any instant, is given by the solution of this equation (I.).

Let us write equation (I.) in the form

$$\frac{d^2 q}{d t^2} + a \frac{dq}{dt} + b q = 0.$$

In order to solve this we may proceed as follows:—The charge q in the jar begins by possessing a certain initial value, and ends by being zero. Let us assume that q can be expressed as a function of the time t in the form $q = A e^{mt}$, when A is a constant and e is the base of the Naperian logarithms, and m is also a certain function determined by the capacity resistance and inductance of the system. For it is clear that by a suitable value for A and m the function $A e^{mt}$ may be made to express the mode in which the charge q dies away with increase of the time t. The problem is reduced, then, to finding A and m. The solution of every differential equation is always by a process of happy guessing; there is no systematic or direct method of obtaining the required result. Take, then, the expression $q = A e^{mt}$, and obtain the first and second differential coefficients, and substitute these results on the original equation (I.), we obtain the expression

$$A e^{mt} (m^2 + a m + b) = 0.$$

Hence, the value $A e^{mt}$ assumed for q will satisfy the equation (I.); that is, when substituted for q in the original equation, render it zero provided that m is such a quantity that $m^2 + a m + b = 0$. The two roots of this last quadratic equation are obtained by a simple solution, and they are

$$m = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}.$$

Two cases then arise, *first*, when $\frac{a^2}{4}$ is greater than b—that is, when $\frac{R^2}{4 L^2}$ is greater than $\frac{1}{LC}$, or $\frac{R^2}{4 L}$ greater than $\frac{1}{C}$. In this case the roots of the quadratic are *real*, and if we call them m_1 and m_2 we can say that the solution of the differential equation (I.) is

$$q = A e^{m_1 t} + B e^{m_2 t} \quad . \quad . \quad . \quad . \quad (II.)$$

where A and B are constants determined by the initial circumstances of the discharge, and m_1 and m_2 are equal respectively to $-\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}$ and $-\frac{a}{2}-\sqrt{\frac{a^2}{4}-b}$. This solution for the value of q is called an exponential solution, and it indicates that under these circumstances when the inductance, resistance and capacity are of such magnitudes that R is *greater* than $\sqrt{\frac{4\,L}{C}}$, the quantity q dies away regularly, diminishing with the time in a continuous manner. In this case the discharge of the jar is always in one direction, and the current or rate of decay $\left(-\frac{dq}{dt}\right)$ of the charge is also always in one direction. If, however, R is *less* than $\sqrt{\frac{4\,L}{C}}$, then $\left(\frac{a^2}{4}-b\right)$ is a negative quantity, and the square root of it is an imaginary one, and the roots of the quadratic $m^2 + a\,m + b=0$ are unreal. It is shown in treatises on algebra that a quadratic equation has either two real or two imaginary roots, and when this last is the case the roots of the quadratic can always be expressed in the form $\alpha + \beta\sqrt{-1}$.

Accordingly, the solution of the original equation (I.) under these circumstances is of the form

$$q=A\,e^{(\alpha+\beta\sqrt{-1})t}+B\,e^{(\alpha-\beta\sqrt{-1})t} \quad . \quad . \quad \text{(III.)}$$

By a simple transformation, based on the employment of the exponential values of the sine and cosine, as given on page 106, this solution can be thrown into the form

$$q=e^{\alpha t}\,(P\cos\beta t+P^1\sin\beta t) \quad . \quad . \quad . \quad \text{(IV.)}$$

where P and P^1 are constants, and

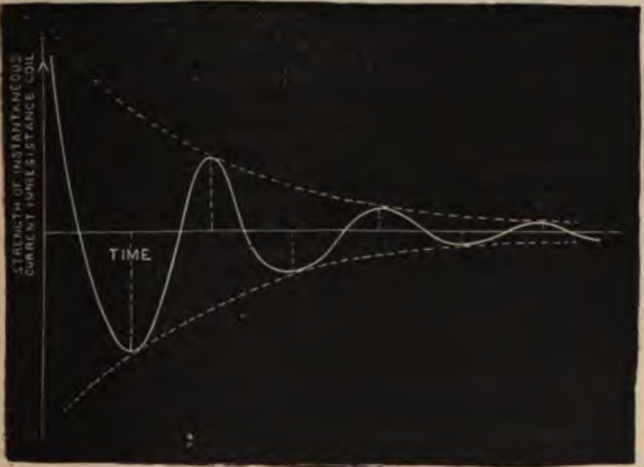
$$\alpha=-\frac{a}{2}=-\frac{R}{2\,L}, \text{ and } \beta=\sqrt{b-\frac{a^2}{4}}=\sqrt{\frac{1}{L\,C}-\frac{R^2}{4\,L^2}}.$$

The general result is then that the equation

$$\frac{d^2q}{dt^2}+\frac{R}{L}\,\frac{dq}{dt}+\frac{1}{LC}\,q=0$$

has two solutions—one, which applies when R is greater than $\sqrt{\frac{4L}{C}}$, and is of an exponential form, and indicates that the

Curve representing the Discharge of a Condenser through a Large Resistance. Discharge Unidirectional and Continuous.



Curve representing the Discharge of a Condenser through a Small Resistance. Discharge is Periodic and Alternate. Maxima gradually diminishing in Geometric Progression.

Fig. 9.

charge q dies away regularly with lapse of time, and the discharge current is uni-directional; the other, which applies when R is less than $\sqrt{\frac{4\,L}{C}}$, contains sine and cosine terms, and indicates a periodically changing discharge decreasing by a series of oscillations, in which the charge in the condenser is first positive and then negative, but at the same time decreasing; or, in other words, is a periodic variation superimposed on a steadily decreasing variation, the currents or rates of discharge following the same distinction. These two modes of discharge, or solutions of the differential equation, are best indicated graphically by the two curves in Fig. 9, in which the upper curve represents the gradual decrease, according to an exponential law, which is indicated as the proper solution of the equation, when the value of R or the resistance of the discharging circuit is greater than $\sqrt{\frac{4\,L}{C}}$, and the lower one the oscillatory discharge, which is indicated by the trigonometrical solution of the differential equation, when the resistance R is less than $\sqrt{\frac{4\,L}{C}}$. When R has such a value that $R=\sqrt{\frac{4\,L}{C}}$, the discharge is just non-oscillatory. We find, then, that according to Sir W. Thomson, analysis indicates that for a certain relation between the resistance and inductance of the discharge circuit and of the capacity of the jar the discharge is a simple current in one direction or a dying-away-backwards-and-forwards current, according as R is greater or less than $\sqrt{\frac{4\,L}{C}}$. If the discharge is oscillatory then the electrical oscillations are isochronous, and the periodic time of a complete oscillation is

$$T=\frac{2\,\pi}{\sqrt{\frac{1}{LC}-\frac{R^2}{4\,L^2}}},$$

for in the second solution (IV.),

$$q=e^{\alpha t}(P\cos\beta t+Q\sin\beta t),$$

we see that at intervals of time equal to $\frac{\pi}{\beta}$ the sine and cosin terms have the same values, since $\sin\beta t=\sin\beta\left(t+\frac{\pi}{\beta}\right)$, an the same for the cosine. Hence, the trigonometrical factor in the value for q periodically repeats itself in value at intervals of time equal to $\frac{\pi}{\beta}$, and is zero at times when $\tan \beta t = -\frac{P}{Q}$. Hence, the complete periodic time of the oscillation is

$$\frac{2\pi}{\beta} \quad \text{or} \quad \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}.$$

and the frequency of the oscillations, or number in one second, is

$$\frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Accordingly, when $R = \sqrt{\frac{4L}{C}}$ there are no oscillations in one second, or the motion is just non-oscillatory, or *dead beat.* In the case of the uni-directional discharge the values of the instantaneous current in the discharge circuit can be represented as we have seen by the ordinates of an exponential curve, and in the case of the oscillatory discharge by those of a periodic curve whose maxima descend in geometric progression as the time increases in arithmetic progression. During equal intervals of time the whole quantities which pass decrease also in geometric progression, and the zero points, or instants of reversals of sign of current, are uniformly separated.

The foregoing predictions of analysis have been confirmed by the experiments of Feddersen, Paalzow, Bernstein, Blaserna, Helmholtz, Schiller and Rood. Thomson in his original Paper pointed out and suggested the application of Wheatstone's mirror in the examination of the discharge. In Feddersen's experiments the spark from a Leyden jar battery was taken between two brass balls placed in front of a revolving mirror. The discharge was passed through a high resistance. The image of the spark was viewed by a telescope. Under these circumstances the image of the spark was drawn out when the mirror revolved into a continuous band of light in a direction perpendicular to that of the discharge.* When the resistance

* An experimental research of a very complete character on the duration and nature of the discharge of a Leyden jar is described by Prof. Ogden

was gradually reduced a point was reached at which the image was broken up into a series of separated strips, each strip corresponding to a discharge. This showed that the discharge was intermittent.

In Paalzow's experiments a similar discharge from a Leyden battery was passed through a resistance coil and through a vacuum tube, and the image of the discharge in the vacuum tube viewed in a revolving mirror. As before, with a small resistance the image consisted of a number of separate images, each of which corresponded to a discharge, and a bluish light showed itself at *both* poles of the vacuum tube. When the resistance was increased the bluish light showed itself only at one pole. In the former case a magnet held outside the tube split the discharge into *two* lines of light, showing that it consisted of currents travelling in both directions; but in the last case the magnet did not divide the discharge. This sufficiently indicated that with a low resistance the discharge was oscillatory and alternate, and not uniform or uni-directional.

Feddersen found that the critical resistance at which the discharge just becomes oscillatory varies inversely as the square root of the capacity of the battery, which is in agreement with the predictions of theory.

A good account of the researches of these experimentalists is given in Wiedemann's *Galvanismus*, Part II., § 800, *et seq.**

We can cast the expressions for the charge at any instant left in the condenser into more convenient forms. *First*, consider the *dead-beat* case (Equation II.) is

$$q = \mathrm{A}e^{m_1 t} + \mathrm{B}e^{m_2 t},$$

Rood in the *American Journal of Science and Arts* for September, 1869; January, 1871; September, 1871; October, 1872; November, 1872; March, 1873. The author's attention was drawn to these Papers by Mr. W. H. Snell.

* For the sake of readers wishing to pursue the subject we give here the references, to which are added some collected by Mr. Tunzelmann in a series of articles on Electrical Oscillations in *The Electrician* of September 14, 1888, and succeeding numbers.

Feddersen, *Poggendorff's Annalen*, Vol. CIII., p. 69, 1858; Vol. CVIII., p. 497, 1859; Vol. CXII., p. 452, 1861; Vol. CXIII., p. 437, 1861; Vol. CXV., p. 336, 1862; Vol. CXVI., p. 132, 1862.

Paalzow, *Pogg. Ann.*, Vol. CXII., p. 537, 1861; Vol. CXVIII., p. 178, 1863.

Bernstein, *Pogg. Ann.*, Vol. CXLII., p. 54, 1871.

Helmholtz, *Monatsberichte der Berl. Akad.*, 1874. *Footnote continued.*

where m_1 and m_2 are the real roots of the quadratic equation

$$m^2+am+b=0;$$

and as $a=\frac{R}{L}$ and $b=\frac{1}{CL}$, we have $m_1=-\frac{R}{2L}+\sqrt{\frac{R^2}{4L^2}-\frac{1}{CL}}$, which we will write as $-\alpha+\beta$, and similarly, m_2 is $-\alpha-\beta$.

The constants A and B are determined by the condition that when $t=0$ the charge q is the original charge Q;

hence $$Q=A+B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (V.)$$

and since the current i at any instant is the rate of loss of charge, or $-\frac{dq}{dt}$, we have $i=-\frac{dq}{dt}=-Am_1e^{m_1t}-Bm_2e^{m_2t}$, when $t=0$, $i=0$.

Hence $$Am_1+Bm_2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (VI.)$$

From these two equations (V.) and (VI.) A and B are determined in terms of m_1 and m_2, or of α and β, and we find

$$A=\frac{\alpha+\beta}{2\beta}Q,$$

$$B=-\frac{\alpha-\beta}{2\beta}Q.$$

Let the quantity $\frac{1}{\alpha-\beta}$ be called T_1 and let $\frac{1}{\alpha+\beta}$ be called T_2, then it is easily seen that $A=\frac{T_1}{T_1-T_2}Q$, and $B=-\frac{T_2}{T_1-T_2}Q$,

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Kirchoff, *Gesammelte Abhandlungen*, p. 168, containing remarks and criticisms of Feddersen's results.

Von Oettingen, *Pogg. Ann.*, Vol. CXV., p. 115, 1862; also Jubelbaud, p. 269, 1874.

L. Lorenz, *Wiedemann's Annalen*, Vol. VII., p. 161, 1879.

Schiller, *Pogg. Ann.*, Vol. CLII., p. 535, 1872.

Mouton, Thèse, Paris, 1876, *Journal de Physique*, Vol. VI., pp. 5 and 46, 1876.

Kolacek, *Beiblätter en Wiedemann's Annalen*, Vol. VII., p. 541, 1883.

Olearsky, *Verhandlungen der Academie von Krakau*, Vol. VII., p. 141, 1882.

Oberbeck, *Wiedemann's Annalen*, Vol. XVII., pp. 816—1040, 1882; Vol. XIX., pp. 213 and 265, 1883.

Bichat et Blondlot, *Comptes Rendus*, Vol. XCIV., p. 1590, 1882.

and the equation for q may be written

$$q=\frac{Q}{T_1-T_2}\left\{T^1 e^{-\frac{t}{T_1}}-T_2 e^{-\frac{t}{T_2}}\right\}. \quad . \quad \text{(VII.)}$$

The ratio of the potential v of the condenser at any instant to its original potential V is the same as that of q to Q.

The two quantities T_1 and T_2 are such that their sum is equal to C R and their product to C L—statements easily verified by taking the values of T_1 and T_2 in terms of α and β, and recollecting that α stands for $\frac{R}{2L}$, and β for $\sqrt{\frac{R^2}{4L^2}-\frac{1}{CL}}$.

Hence also the current i at any instant is given by

$$i=\frac{Q}{T_1-T_2}\left\{e^{-\frac{t}{T_2}}-e^{-\frac{t}{T_1}}\right\} \quad . \quad . \quad \text{(VIII.)}$$

These two equations (VII.) and (VIII.) contain the complete solution of the discharge in the *dead-beat* case, giving the current, potential and quantity at any instant reckoned from the moment of closing the circuit of the condenser.

Suppose that the discharging circuit possesses no inductance, then $L=0$, and the equation (VII.) reduces to

$$q=Q e^{-\frac{t}{RC}}.$$

In the above the product R C, or the product of the resistance of the discharging circuit and capacity of condenser, is a quantity of the *dimensions* of a time, and is called the *time constant* of the condenser. It represents the time in which the charge of the condenser falls to $\frac{1}{e}$th part of its original value (e being 2·71828). Let R C be denoted by T. Then, if we begin with a charge Q in a time T the charge left is $\frac{Q}{e}$. In a time 2 T it is $\frac{Q}{e^2}$, and in a time n T it is $\frac{Q}{e^n}$. Now, since $e^3=(2{\cdot}71828)^3$, or nearly 20, and e^4 is nearly 54, it follows that in a time 7 T only one-thousandth of the original charge remains, and in a time 21 T only one thousand-millionth; so that in a period of time equal to 5 or 6 times the length of the time constant the condenser is practically discharged. If the discharging circuit possesses inductance then in the dead-beat case there are two time constants of unequal importance. These are the quantities we have called T_1 and T_2 above. T_1 is the larger of the two. The rapidity of decay of the charge with an inductive discharger depends chiefly on T_1. For if we refer again to equation VII., we see that q will become zero when the quantity in the bracket, viz., the function $\{T_1 e^{-\frac{t}{T_1}} - T_2 e^{-\frac{t}{T_2}}\}$, becomes zero.

Now, starting with given values of T_1 and T_2 depending on the values of L, C, and R, and having that T_1 is greater than T_2, the function starts with a value equal to $T_1 - T_2$ when $t=0$, and as t increases without limit both exponentials tail away down to zero; but since T_1 is greater than T_2, the first exponential, viz., $e^{-\frac{t}{T_1}}$, is longer getting down to practical zero than the other. Hence, the evanescence of $e^{-\frac{t}{T_1}}$ practically determines the time of discharge of the condenser, and we can call T_1 the principal time constant of the system.

If we call the expression $\frac{L}{CR^2}$ λ, then bearing in mind that

$T_1 = \frac{1}{\alpha - \beta}$ and $T_2 = \frac{1}{\alpha + \beta}$, where $\alpha = \frac{R}{2L}$ and $\beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$, we can express T_1 and T_2 in terms of λ and C R or T, and we have by simple substitution

$$T_1 = \frac{2T\lambda}{1 - \sqrt{1-4\lambda}},$$

and

$$T_2 = \frac{2T\lambda}{1 + \sqrt{1-4\lambda}},$$

and the product $T_1 T_2 = T^2 \lambda$.

Hence, if a horizontal line is taken, on which values of λ are set off (see Fig. 10), and values for T_1 and T_2 plotted off vertically, the locus of the extremities of these ordinates is a parabola. In the figure, lengths along O 1 represent values of λ, and the corresponding values of T_1 and T_2 define a parabola

P M O, such that O P = T = C R, and the ordinates of the upper portion P M of the curve are the values of T_1, and those of O M are those of T_2. The value of $\lambda = \frac{1}{4}$ is the abscissa O A, for which $T_1 = T_2$, for when $\frac{L}{C R^2} = \frac{1}{4}$ then $\beta = 0$, and in this case $T_1 = T_2$, and T_1 has its minimum value. For this particular value of λ, which is just the value when the discharge ceases to be dead beat, and becomes oscillatory—that is, when $\frac{R^2}{4 L^2} = \frac{1}{C L}$

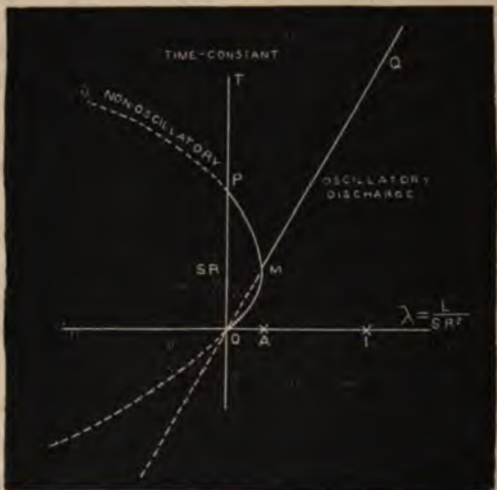

FIG. 10.

or $\frac{L}{C R^2} = \frac{1}{4}$—the time constants have equal values, and T_1 becomes a minimum. Hence, for this particular value of the inductance the time of discharge of the condenser is a minimum, and *less*, therefore, than the time of discharge when the discharge circuit has no inductance.*

* This appears to have been first noticed by Mr. W. E. Sumpner (*Phil. Mag.*, June, 1877), and discussed by Prof. Oliver Lodge in an interesting paper in *The Electrician* for May 18, 1888, p. 39, from which article some portion of the above paragraph and figures has been taken.

Turning next to the case when the inductance of the discharge circuit is such that λ is greater than $\frac{1}{4}$, or when $\frac{R^2}{4L^2}$ is less than $\frac{1}{CL}$, we have to consider the periodic function which then applies.

Referring to equation IV. for the value of q in terms of t we have

$$q = e^{\alpha t} (P \cos \beta t + P^1 \sin \beta t),$$

where $\alpha = -\frac{R}{2L}$ as before, but β now stands for $\sqrt{\frac{1}{CL} - \frac{R^2}{4L^2}}$.

From the conditions that $q = Q$ when $t = 0$, and that when $t = 0, i = \frac{dq}{dt} = 0$, we find that $P = Q$ and $P^1 = Q\frac{\alpha}{\beta}$.

Hence, $$q = Q e^{-\frac{Rt}{2L}} \left\{ \cos \beta t + \frac{R}{2 \beta L} \sin \beta t \right\}.$$

On the convention that γ is such an angle that

$$\tan \gamma = \frac{2 L \beta}{R},$$

we can write the above expression

$$q = Q e^{-\frac{Rt}{2L}} \left\{ \frac{\sin (\beta t + \gamma)}{\sin \gamma} \right\},$$

and

$$i = \frac{dq}{dt} = \frac{Q}{\beta L C} e^{-\frac{Rt}{2L}} \sin \beta t.$$

Hence, we see that the expression for the currents and for the remanent quantity of electricity at any time t consists of a periodic part, which is a sine function, and a die-away part, which is an exponential function, and that the rate of decay of the maxima of the waves is determined by the value of $\frac{R}{2L}$; in other words, $\frac{2L}{R}$ is the time constant for the oscillatory form of discharge.

This is expressible as $2 T \lambda$ in our notation, and is, hence, simply proportional to λ. In Fig. 10 the variation of the time constant T_3, or $\frac{2L}{R}$, for oscillatory discharge is represented by the straight line M Q.

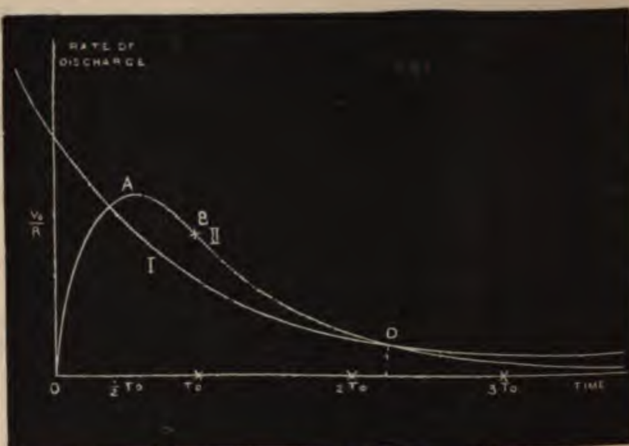

FIG. 11.

Curve I. represents the strength of the discharge current of a condenser in a circuit of no self-induction. $T_0 = S R$. This curve corresponds to the point P in Fig. 10.

Curve II. represents the strength of the discharge current of the same condenser in a circuit of the same resistance, but with self-induction enough just to bring the discharge to the verge of oscillation, this being the condition which effects complete discharge in the shortest time possible. This curve corresponds to the point M in Fig. 10.

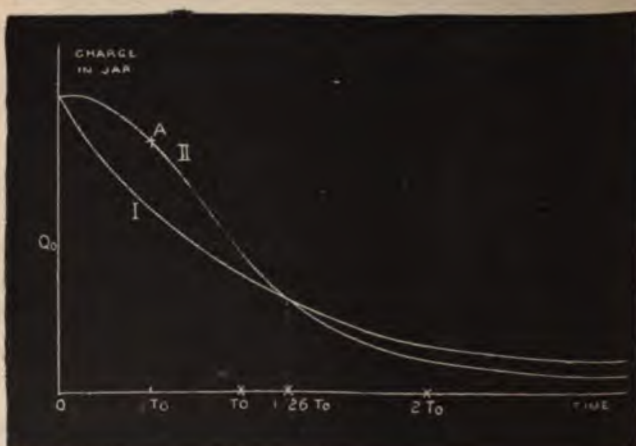

FIG. 12.

Curve I. shows the charge remaining in the jar at any time, the circuit being practically devoid of self-induction.

Curve II. shows the same thing for $L = \frac{1}{4} S R^2$—that is, for the quickest discharge possible. At first Curve I. has the advantage, but at a time 1·26 R S the second curve overtakes it and discharges the jar more rapidly.

The really important part of the time constant curve is the part P M Q, consisting of a bit of a parabola and a straight line, and having a minimum ordinate corresponding to $\lambda = \frac{1}{4}$.

The current at different times for the two cases $\lambda = 0$ and $\lambda = \frac{1}{4}$ are plotted in Figs. 11 and 12.

For $\lambda = \frac{1}{2}$ we have $T_3 = T$, since $T_3 = 2\,T\,\lambda$. In other words, the time of discharge of the condenser when $\frac{L}{C\,R^2} = \frac{1}{2}$ is the same as when $L = 0$, and just *double* that when $\lambda = \frac{1}{4}$; and in this last case the rate of discharge is a maximum. Hence, so far from reducing the rate of discharge, a *little* self-induction in the discharge circuit is a positive help to the condenser in getting rid of its charge. Mr. Sumpner* has pointed out that since a lightning discharge resembles that of a condenser, a little inductance in a lightning rod may assist matters instead of blocking the way of the discharge.

A pendulum swinging in treacle was long ago suggested by Lord Rayleigh as a mechanical analogue to the Leyden jar discharge. Dr. Lodge† has pointed out that we may make the analogy exact by considering a loaded spring bent aside or compressed in a resisting medium in such way that gravity is not concerned in the motion and then let go.

The pliability of the spring corresponds to the capacity of the condenser, its displacement to the electric charge. The load or inertia corresponds to the self-induction of the circuit; the viscosity of the fluid to its resistance. If the viscosity friction be supposed to vary accurately as the speed, then the equation of motion is

$$m\frac{d\,v}{d\,t} - R\,v = R\,x,$$

where x is the displacement and v the velocity $= -\frac{d\,x}{d\,t}$. Writing L for m, and $\frac{1}{R}$ for C, and x for Q, we have the condenser equation (I.); the two are seen to be the same, and everything we have said of the electrical problem applies to the mechanical one.

It is obvious mechanically that if the resistance is moderate

* *Loc. cit.* † See *Electrician*, May 18, 1888, p. 41.

and mass considerable the recoil of the spring will be accompanied by oscillations, and that with great resistance and small inertia the motion will be a slow sliding back without oscillation; and there must exist between the strength of the spring, the mass of its load, and the viscosity resistance of the medium some definite relation which shall constrain the recoil to be dead beat, just returning to the original position of equilibrium without overshooting the mark. This relation is now seen to be

$$R^2 = 4 R m,$$

and under these circumstances the recovery of the spring is effected in the shortest possible time.

In addition to the experimental researches of Blaserna, to which reference has been made at page 199 *et seq.*, very extensive experiments have been made by Bernstein* and by Mouton† on the subject of electrical oscillations of induced currents. Bernstein's experiments were made with a revolving wheel interrupter, which closed a primary circuit, and for a very short time, at a determinable period after closure of the primary, put the secondary circuit in series with a delicate ballistic galvanometer. In this way the state of the secondary circuit could be investigated at various instants of time after closing or opening the primary circuit, and the general results of Blaserna were confirmed. In Mouton's experiments a rather different form of commutator (*see* Jamin's "Cours de Physique," Vol. IV., p. 201; third edition) was employed to break a primary circuit and to examine with a quadrant electrometer the electrical state of the terminals of an open secondary circuit at various instants afterwards. Mouton found that a potential difference declared itself at less than one four-millionth of a second after rupture of the primary, and that this potential difference died away with decreasing amplitude by rapidly reversing sign, thus indicating the existence of electrical oscillations set up in the open secondary circuit. The duration of the first semi-oscillation was greater than that of succeeding ones. In the case of a secondary circuit of 13,860 turns he found that the first semi-oscillation had a duration of 110

* *Pogg. Ann.*, Vol. CXLII., p. 54, 1871.

† "Étude expérimental sur les phénomènes d'induction électrodynamique." Thése de Doctorat 1876.

millionths of a second, and the succeeding ones about 77 millionths of a second, and he was able to count about thirty complete oscillations.

§ 7. The Function of the Condenser in an Induction Coil.—Fizeau appears to have been the first* to suggest that the action of an induction coil employed for raising the electromotive force of a current would be increased by the employment of a condenser. The mode of use is as follows:—Let P be a primary circuit which takes current from a few cells of a battery, and let I be an interrupter in the primary circuit, either automatically worked by the magnetisation and demagnetisation of the iron core or by any other means. Let S be a secondary circuit of many more turns and high resistance. Under these circumstances each break of the primary current is accompanied by the production of an electromotive force in the secondary capable of producing a discharge across an air space in the secondary circuit. This electromotive force in the secondary is increased by any action tending to increase the suddenness of the stoppage of the primary current, and decreased by anything promoting a spark at the points of rupture of the primary circuit. Fizeau found that if a condenser, formed of alternate sheets of tinfoil and mica or paraffined paper in such fashion as to form a Leyden jar, has its two opposite coatings connected with the two extremities between which the rupture of the primary circuit takes place, then the electromotive force in the secondary circuit under the same circumstances is increased. In most current text-books and elsewhere this action is explained by saying that the extra current in the primary circuit, instead of being expended in making a spark at the contact points, darts into the condenser and hastens the decay of the primary current. This explanation as generally given is, however, very imperfect. A more complete examination of the nature of the condenser action has been given by Lord Rayleigh (*Phil. Mag.*, Vol. XXXIX., 1870, p. 428, *et seq.*). In the experiments there detailed a sewing needle was submitted to the magnetising action of an induced secondary current produced by the "break" of current in a primary circuit. In some previous experiments

* *Comptes Rendus*, Vol. XXXVI., p. 418, 1853.

by the same writer (*Phil. Mag.*, July, 1869, p. 9) it had been shown that the magnetising effect of the secondary current was *cet. par.* proportional to the *initial* strength of the induced current, and that this initial strength was proportional to the quotient of M by N, or to the value of the ratio of the coefficient of mutual induction to the coefficient of self-induction of the secondary circuit. It was then found that the magnetising effect of the secondary current was greatly increased by connecting the plates of a condenser respectively to the two points between which the break of the primary circuit occurred. The complete investigation of the values of the induced and primary currents would under these conditions be a good deal more complicated than the investigation of the more simple case of discharge of a condenser through a single inductive circuit. We are here, however, only concerned with the first part of the electrical motion, the manner in which the currents wear down under the action of the resistances being of subordinate importance. It appears that when the electrical motion is decidedly of the oscillatory type the first few oscillations will take place almost uninfluenced by resistance, and on this supposition the calculation (following Lord Rayleigh) becomes remarkably simple.

Let L, M and N be the primary, mutual and secondary inductance, and let i and i' be the primary and secondary current strengths at any instant, and q and q' the quantities of electricity which have flowed through these circuits from the instant of beginning to reckon the time t,

then $$\frac{dq}{dt}=i \text{ and } \frac{dq'}{dt}=i';$$

and if we neglect resistance effects, as we can do at the instant after "breaking" the primary circuit, and call C the capacity of the condenser bridging across the "break" of the primary circuit, the equations giving the values of the primary and secondary current i and i' at the instant after breaking the primary circuit are—

$$L\frac{di}{dt}+M\frac{di'}{dt}+\frac{q}{C}=0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$M\frac{di}{dt}+N\frac{di'}{dt} \qquad =0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating i' we have

$$\left(L-\frac{M^2}{N}\right)\frac{di}{dt}+\frac{q}{C}=0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(3) may be written

$$\left(L-\frac{M^2}{N}\right)\frac{d^2q}{dt^2}+\frac{q}{C}=0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

A differential equation of this type always indicates an oscillatory motion. For consider the simple periodic function, $x = A \sin n t$, where $n=\frac{2\pi}{T}$, T being the periodic time of the motion, we have $\frac{dx}{dt}=n A \cos p t$, and $\frac{d^2x}{dt^2}= -n^2 A \sin p t$; hence, $\frac{d^2x}{dt^2}+n^2 x=0$, and therefore $x=A \sin p t$ is a particular solution of this equation.

In the above differential equation n is seen to be 2π times the frequency of the oscillation.

Accordingly, equation (4) indicates an oscillation of the primary current, of which the periodic time is equal to

$$2\pi\sqrt{C\left(L-\frac{M^2}{N}\right)},$$

and this is the periodicity of the electric oscillation set up in the primary at the first instant after "break."

Equation (2) gives by integration the connection between i and i', and it is

$$M i+N i'=\text{constant} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which shows that the currents in the primary and secondary oscillate synchronously, the maximum of the one coinciding with the minimum of the other. Since i' is zero at the instant of "break," the constant in equation (5) must be equal to M I, where I is the current strength in the primary just before "breaking" primary circuit.

Accordingly, we have

$$i'=\frac{M}{N}(I-i),$$

so that when, after half an oscillation of the primary, i becomes equal to $-$ I, we have

$$i' = 2\frac{M}{N}I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation gives us the *initial* value of the secondary current i' in terms of the value of the primary current just before the "break" when the condenser is used. Comparing equation (6) with the results on page 185, where it is shown that, if the condenser is not used across the "break" of the primary, the initial value of the secondary current under the assumption of a perfectly sudden break is equal to $\frac{M}{N}I$, we see that the value of the secondary current, just immediately after the break of the primary, is *double* that which is there deduced as the value when the primary is simply suddenly stopped without the intervention of the condenser. Stripped of symbolism, what the above amounts to is this: if a condenser is inserted across the "break points" of a primary circuit, then on breaking the primary current continues to run on into the condenser for a little bit; it then *rebounds*, and is reversed in sign, retaining initially its full strength. The electromotive force set up in the secondary circuit is then the result of a stoppage of a primary current and its *immediate reversal* in direction, and this is equivalent to the removal of a certain number of lines of induction from the secondary circuit, and their immediate insertion into it in the opposite direction. Hence, when a condenser is so employed the inductive electromotive force in the secondary must be just double that which it would be if there were no such rebound of the primary. The condenser acts by setting up electrical oscillations, and it does away with the spark, or largely diminishes it, in virtue of the fact that the condenser acts at the moment of "break" as if it were a shunt circuit of negative self-induction, only with this difference—that instead of dissipating energy like a conducting circuit it returns it again to the primary circuit in the form of a reversed current, and increases the total change of induction through the secondary circuit in the short interval of time immediately succeeding the "break."

Since the sparking distance of the secondary current depends on the initial electromotive force in the secondary—that is, on the maximum of the electromotive force—we see that the condenser so applied can greatly increase the sparking distance of the secondary discharge.

The action is essentially a phenomena of *resonance*. The condenser causes an elastic recoil in the current and enables the electro-kinetic energy of the steady primary current to be utilised in producing secondary electromotive force rather than suffer dissipation in the form of a contact spark. In order to be efficient in quenching spark the capacity of the condenser must be great enough to take up the full primary current, or to receive charge at a rate equal to the delivery of the full primary current for a time during which the contact or break points are separating to a distance too great to permit of much sparking jumping across. There is a certain capacity of condenser suitable for any given coil which produces the most beneficial result in quenching contact spark and lengthening secondary spark. The required capacity is best determined by trial, since the experimental data necessary to furnish the means to calculate it would be probably more difficult to obtain, owing to the fact that it will be determined by several variables, viz., the effective resistance and inductance of the primary circuit, the rate of breaking, and probably also by the amplitude of movement of the "break points." If the primary coil of an induction coil is traversed by an alternating current then the condenser as ordinarily used becomes superfluous. It will be remembered that the late Mr. Spottiswoode obtained secondary sparks of great magnitude from his large coil by so using the alternating current of a De Meritens machine.

If a condenser is discharged through a circuit of which the resistance is so small that it may be neglected in numerical comparisons, then the equation of discharge is

$$L\frac{d^2 q}{d t^2}+\frac{q}{C}=0,$$

where the symbols have the same signification as before. As above explained, this indicates that the discharge is oscillatory, and that the time of a complete oscillation is $2\pi\sqrt{LC}$.

In describing the experiments of Blaserna we saw that the frequency of the electrical oscillation set up in circuits on starting and stopping currents in them could be reckoned by tens of thousands per second. In the case of Leyden jars discharged through very short circuits, the frequency may rise to numbers reckoned by millions per second. Since the frequency of luminous vibrations falls between 400 and 700 billions per second, these condenser oscillations fall in frequency in the gap between the acoustic and luminous vibrations.

It is of interest here to note that since these electrical oscillations in a circuit are creating pulsatory electrical disturbances, which spread out from the wire laterally, the wire in which the electrical oscillations are going on is virtually emitting "light," although not such light as can affect our eyes. The ether waves in the case of these electrical disturbances are too long to be eye-affecting. If the velocity of a wave disturbance is V, and the wave length is λ, and the frequency of the oscillations corresponding to this wave length is n, then $V = n\lambda$, for the wave motion travels over the length of one wave in the time of one complete oscillation. In the case of ether disturbances we have seen that V is 3×10^{10} centimetres per second, or 186,000 miles per second. Hence when the frequency of the electrical oscillations is known, the wave length of the lateral disturbance emitted can be found. According to Dr. Lodge, a microfarad condenser discharging through a good conducting coil of one secohm self-induction gives a current alternating 160 times in a second, and emits ether waves about 1,200 miles long. A gallon Leyden jar (capacity about ·003 microfarad) discharging through a stout wire suspended round an ordinary sized room emits ether waves between three and four hundred yards in length, its current alternating at the rate of about one million per second. A pint Leyden jar sparking through an ordinary pair of discharging tongs gives a current of 15 million alternations per second, with ether waves some 20 yards in length. An ordinary electrostatic charge on a sphere two feet in diameter, if disturbed in any way, will surge to and fro at the rate of 300 million vibrations per second, emitting ether waves a yard long. Electric charges on bodies of atomic dimensions, if able to oscillate at all, would vibrate thousands of billions of times a second, and produce ultra-violet light.

The ordinary use of a condenser with an induction coil shows how it can be employed to neutralise the effect of self-induction in a circuit. There is another very practical case of a like nature. In the employment of a *relay* or electro-magnetic circuit-closer in telegraphy the self-induction of the relay is an obstacle to the production of rapid changes of current strength through the relay. In this case it is found that if the terminals of the relay are joined to the poles of a suitable condenser the effective self-induction of the relay is thereby lessened. Let L_1 R (Fig. 13) be an inductive circuit, and let the terminals $a\,b$ be closed by a condenser C of capacity C. Let L be the inductance and R the resistance of the coil. Let i be the value at any instant of a simple periodic current sent through the relay and condenser in parallel, and let i_1, i_2 be the simultaneous current strengths at that instant in the condenser circuit and

FIG. 13.

the coil circuit. As the potential difference of the points a and b oscillates there is produced in the condenser circuit an ebb and flow of current; the condenser, in fact, is charged and discharged by the periodic current; also a periodic current is produced in the inductive circuit L_1 R. The current in L_1 R lags in phase behind the impressed electromotive force or potential difference of the points $a\,b$, and the current flowing into the condenser lags 90 deg. in phase behind the same impressed electromotive force. From this it results that the mean current through the inductive circuit may, under some circumstances, be *greater* when the condenser is joined up to its ends than when it is not so joined; its effective self-induction is thereby lessened, and it acts as if it had experienced a diminution of self-induction. The condition most favourable for producing this result may be investigated as follows:—

Let v be the potential difference of the points a and b at the instant when the current in the undivided circuit is i and that in the branches is i_2 and i_1. We then have, by the principle of continuity,

$$i = i_1 + i_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

also

$$i_1 = C \frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$L \frac{di_2}{dt} + R i_2 = v. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and we may take the original current before division to be simply periodic, and to be expressed by

$$i = I \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where I is its maximum value.

Then by elimination of v and i_1 and i from the above four equations we arrive easily at the equation—

$$C L \frac{d^2 i_2}{dt^2} + C R \frac{d i_2}{dt} + i_2 = I \sin pt \quad . \quad . \quad . \quad (5)$$

Now, since i_2 must be a simple periodic current lagging in phase behind that of the undivided current i, we may take i_2 to be of the form

$$i_2 = I_2 \sin (pt - \theta) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

I_2 being the maximum value of i_2, and θ its phase lag behind I_1.

Hence, by differentiation of (6) and substitution in (5) we arrive at

$$(1 - C L p^2) I_2 \sin (pt - \theta) + C R p^2 I_2 \cos (pt - \theta) = I \sin pt \quad (7)$$

which by the lemma on page 132 may be written—

$$I_2 \sqrt{(1 - C L p^2)^2 + C^2 R^2 p^2} \sin (pt - \theta + \phi) = I \sin pt \quad . \quad (8)$$

Both sides of this last equation are the expressions for the same thing, viz., the value of i, and hence, equating the coefficients, we have

$$\left(\frac{I}{I_2}\right)^2 = (1 - C L p^2)^2 + C^2 R^2 p^2 \quad . \quad . \quad . \quad (9)$$

This gives us the value of the ratio of the maximum or mean values of the strengths of the undivided current and the current in the inductive circuit. If we differentiate the right-hand side of (9) with respect to C, and apply the usual criterion to ascertain whether we have a maximum or minimum value, we find that the expression on the right-hand side of (9) has a minimum value when

$$C = \frac{L}{R^2 + p^2 L^2} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In other words, if the capacity of the condenser is so chosen as to have a capacity equal numerically to the quotient of the inductance by the impedance of the coil, then under these circumstances the mean strength of the current in the coil circuit will be *greater* than the mean strength of the current before subdivision; and it is easily seen by substituting in equation (9) the value of C given by (10), which makes the ratio of current strength a minimum, that with this value of the capacity strength of the current in the inductive coil is to the strength of the current before division in the ratio of the impedance to the resistance of the inductive circuit.

The expression (10) gives the value of the condenser capacity which will produce the required result of minimising the self-induction of a relay of resistance R and inductance L when applied to it. Another problem of a like kind, but not so practically useful, is the investigation of the behaviour of a condenser when joined in *series* with an inductive coil and traversed by a simple periodic current. Let a condenser of capacity C be joined in series with an inductive circuit of resistance R and inductance L, and let a simple periodic current of frequency n be sent through the two in series. It is not difficult to show that, if we take p for $2\pi n$, as usual, and if the capacity and inductance are so related to the frequency of oscillation that $p = \frac{1}{\sqrt{LC}}$, then, under these circumstances, the condenser just annuls the self-induction of the coil, and the two together permit the passage of the same current which would traverse the coil in virtue of its resistance R, assuming it to have no inductance. We leave the proof of this as an exercise to the student.*

§ 8. **Impulsive Discharges and Relation of Inductance thereto.**—If between the ends of a conductor a difference of potential is created which is brought about slowly, the result shows itself in a current in the conductor, and the resulting current is determined as to strength by the mode of variation of the potential and by the capacity as well as by the inductance and ohmic resistance of the conductor. If, however, the difference of potential is created with great suddenness, the resulting electric flow is less and less determined by what may be the true resistance, and more by the inductance of the conductor. In this case we have the phenomena of *impulsive discharges*. We have a mechanical analogy in the case of impulses or sudden blows given to heavy bodies, which well illustrates how strikingly force phenomena may be altered when for steady or slowly varying forces we substitute exceedingly brief impulses or blows. If an explosive, such as gun-cotton, is laid on a stone slab in open air and simply ignited it burns away with comparative slowness, the slab is uninjured, and the evolved gases simply push the air away to make room for themselves. But it is well known that by means of detonators the same explosive can be fired with enormously greater rapidity, and in this case the blow or impulse given to the air is so sudden that it has not time to be pushed away, and in virtue of its inertia its incapability of receiving a finite velocity in an infinitely small time bestows on it an *inertia resistance*, which causes nearly the whole of the effect of the explosion to take effect downwards on the slab, and this last is shattered. The inductance of conductors introduces a series of phenomena which are the electrical analogues of the above mechanical experiment. We have seen that the counter electromotive force of self-induction is proportional to the rate of change of another quantity, called the electro-kinetic momentum, and this quantity physically interpreted is the total flux of induction or number of lines of force enclosed by the conducting circuit at

* Both this and the previous proposition will be found to be proved by a geometrical method in Mr. Blakesley's book on "Alternating Currents," Second Edition.

that instant. A conductor of sensible inductance can no more have a current of finite magnitude created in it instantaneously than a body of sensible mass can have a finite velocity instantaneously given to it. In both cases there is an immense resistance to very sudden change of condition. A very loose plug of snow or earth stuffed into the muzzle of a loaded gun will cause it to burst when fired, since the inertia resistance of the plug to very sudden motion is exceedingly large, though the frictional resistance may be small. Accordingly, the study of the behaviour of conductors under exceedingly sudden electric blows or electromotive impulses leads us to consider some

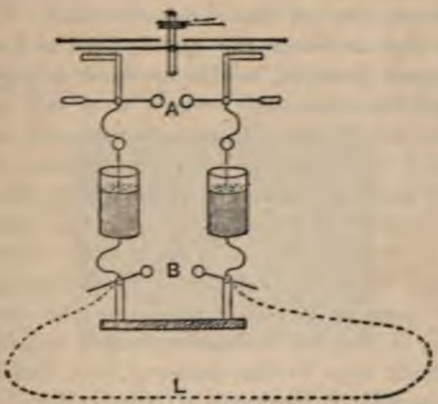

FIG. 14.

very interesting effects. We shall best elucidate these effects by describing some highly interesting and suggestive experiments due to Dr. Oliver Lodge.*

His first experiment is called the experiment of the *alternative path*. The two terminal knobs of a Voss or Wimshurst electrical machine (*see* Fig. 14) are connected to the two inside coatings of a pair of Leyden jars. The two outside coatings are con-

* The account of these experiments is taken from the report of Dr. Lodge's Mann Lectures before the Society of Arts. These delightful and suggestive lectures were reprinted in *The Electrician*, entitled "Protection of Buildings from Lightning," Vol. XXI., pp. 234, 273, 302.

nected to the balls of another discharger, B, and the terminals of this discharger are short-circuited by a metal wire indicated by the dotted line. The Leyden jars stand on a badly insulating wooden base. On turning the handle of the electrical machine the inside coatings receive equal and opposite electrical charges, and there is an induced charge on the outer coating of each, which, in the language of the old school of electricians, was called the "bound" charge. When the difference of potentials of the inner coatings reaches a certain value the air space at A is cracked, and a spark passes, discharging the inner coatings of the jars. At that instant the charges of the outer coatings are set "free," or, in modern language, the potential of one rises and that of the other falls. The effect of this is that whereas before the spark passed at A the balls at B were at equal potential, on the spark at A happening the balls at B are instantaneously brought to a very great difference of potential. It might be thought that since the balls are short-circuited by a metallic wire this difference of potential will expend itself on making a current in the wire. On the contrary, very little of the discharge may take place through the wire. A spark passes at B, or, in other words, the discharge passes in great part across the exceedingly highly resisting air space at B, rather than take the circuit of the metallic wire of very low resistance, so that although there is a divided circuit open to the discharge, one branch of which measures hundreds of thousands of ohms or megohms and the other only a small fraction of an ohm, it nearly all goes by the route of higher resistance. The explanation of this is that when the balls at B are thrown with great suddenness into opposite electrical states the counter electromotive force of self-induction of the circuit of metal L makes it virtually absolutely non-conducting. The electromotive blow meets with such resistance owing to the electro-magnetic inertia of the circuit that it rebounds and cracks through the air. In order that it shall do this, however, the distance of this air gap at B has to be less than a certain amount. There is a certain critical distance of the knobs B for less than which the discharge always jumps across B, and for greater than which the discharge keeps mainly to the metallic circuit. Even if the short-circuiting metal is a thick rod, still when B is not great the discharge chooses the air-gap path. The phenomenon here presented has had fresh interest and attention called to it by Dr. Lodge, but it has really been long a familiar one, though its explanation has not stood out hitherto so sharply as it does now.

Faraday was acquainted with it, and showed that if a charged Leyden jar is discharged by means of a wire crossed or bent so that there was a loop (Fig. 15), the wires at *a* nearly but not quite touching, then when the spark happened at *b* a spark took place also at *a*, showing that some at least of the discharge jumped across *a* instead of pursuing the course of the metal loop.

The same fact lies at the base of the action of all lightning arrestors placed on telegraph or telephone instruments. Mr. C. F. Varley, we believe, first suggested that the coils of the single-needle instrument might be protected from damage by

FIG. 15.

lightning by twisting together the earth and line wires where they leave the case, the theory being that although ordinary currents were not short-circuited by reason of the cotton covering of the wires, yet lightning discharges would meet with such resistance in the inductive coils that they would jump across the knot from wire to wire rather than pass round and damage the coils. In the same way the ordinary *comb protector* is supposed to act. Between the line wire L (see Fig. 16) and the electro-magnetic instrument, relay, telephone, &c., is placed a metal comb, which has its points in opposition to another comb in connection with the earth, and the other terminal of the electro-magnetic instrument is also "to earth." An incoming current has then two paths open to it to get to earth, one of comparatively low resistance through the instrument, and one of enormously high resistance across the air gap between the comb points. Ordinary currents, steady or periodic, pass entirely through the metallic circuit. Very violent electric impulses, such as a lightning discharge, meet with an enormous inductive opposition in the electro-magnetic instrument owing to the inability of an inductive circuit to respond to an electromotive impulse instantaneously. Hence the air gap is cracked, and the discharge passes across the combs and the instrument may be saved. Evidence is, however, abundant pointing to the fact that the protection afforded by these contrivances is very far from complete. We are not here concerned with their efficiency as practical devices, but only with them as illustrating the principle of the alternative path and the

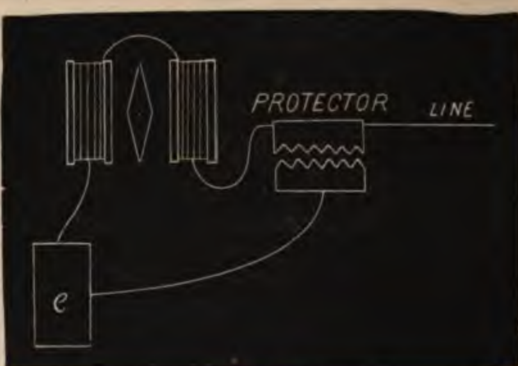

FIG. 16.

behaviour of inductive circuits to impulsive discharges. That these devices are inefficient has been fully demonstrated by many experimentalists.*

In these experiments of the alternative path it was found by Dr. Lodge that the critical distance at which the discharge just prefers to jump the air gap was greater for a thick copper rod 40 feet long (No. 1 B.W.G.) than for an iron wire (No. 27

* For an account of some interesting experiments by Prof. Hughes and Prof. Guillemin on "Lightning Protectors" see *The Electrician*, Vol. XXI. p. 304, July 13, 1888. It was found that a protector consisting of two opposed flat plates was better than a comb or opposed point.

B.W.G.) of 33·3 ohms resistance, indicating a less inductive inertia on the part of the iron; but this fact is only true for the particular circumstances of the experiment. A very clear difference was established between copper rod and tape, using conductors of the same length and weight. The tape has an advantage in permitting more easily the passage of sudden electric discharges. A controversy on the relative suitability of rod and tape for lightning conductors dates from the time of Faraday and Sir W. Snow Harris, and a possible explanation of the reasons for preferring one rather than the other presents itself when we consider the matter in the light of those considerations, which induce us to think that an electric current begins always at the surface of conductor, and takes a certain time to diffuse or soak into the mass of the metal. It is not cross-section but surface which is here concerned; and, other things being equal, the conductor which offers the greatest surface to the dielectric is able to drain the energy out of the dielectric most quickly and dissipate it as heat in the conductor. We have referred to this on a previous page (see *ante*, p. 252), and it will be mentioned again in connection with some views of Prof. Poynting.* With respect to the apparent superiority of iron, it would naturally have been supposed that, since the magnetic permeability of iron bestows upon it greater

* For some special remarks on the self-induction of wires of various cross-sections see Mr. Oliver Heaviside in the *Phil. Mag.*, January, 1887, p. 11:—"The magnetic energy per unit of length of a circuit is $\frac{1}{2}$ L i^2, where i is the current in the wire and L the inductance per unit of length. As regards the diminution of L in general by spreading out the current in a strip instead of concentrating it in a wire, that is a matter of elementary reasoning founded on the general structure of L. If we draw apart currents, keeping the currents constant, thus doing work against their mutual attraction, we diminish their energy at the same time by the amount of work done against their attraction. Thus the quantity $\frac{1}{2}$ L i^2 of a circuit is the amount of work that must be done to take the current to pieces, so to speak—that is, to separate all its filamentary elements of currents to an infinite distance. If wires are taken, each of a unit of length and of the same total cross-sectional area, but of different forms of cross-section, round, square, elliptical, equilateral triangle, narrow rectangle, &c., the ratio of their inductances is the same as the ratio of their torsional rigidities. Thus the narrow strip has the least torsional rigidity, and the circular-sectioned wire the greatest, and this it true also for their relative self-inductions."

inductance, it would form a less suitable conductor for discharging with great suddenness electric energy. Owing to the fact that the current only penetrates just into the skin of the conductor, there is but little of the mass of the iron magnetised, even if these instantaneous discharges are capable of magnetising iron. This last fact has been thought to be due to an actual time lag of magnetisation, viz., that magnetising force required to endure for a sensible time in order to produce magnetisation, but recent views tend in the direction of considering the apparent lag as a consequence of the fact that the eddy currents produced in the surface layers of the metal by the discharge shield the inner and deeper layers from inductive influence, as described under the head of Magnetic Screening. In any event the final result is the same; the electromotive impulses, or sudden rushes of electricity, do not magnetise the iron, and hence do not find in it any greater self-inductive opposition than they would find in a non-magnetic but otherwise similar conductor. Dr. Lodge's further researches seem to show that there is a real advantage in using iron for lightning conductors over copper, and that its greater specific resistance and higher fusing point enable an iron rod or tape to get rid safely of an amount of electric energy stored up in a dielectric which would not be the case if it were copper. This point is further elucidated by some other experiments of Dr. Lodge. Two tinfoil conductors were prepared of approximately equal resistance and length. One of these was formed into a spiral, each layer being insulated with paraffin paper, and wound on a glass tube. The other was made into a zig-zag or non-inductive resistance. These conductors were then employed as alternative paths, as in the former experiment, with the copper wire. In the case when the tinfoil zig-zag was employed to short-circuit the jars it was not possible to get a B spark (*see* Fig. 13) until the distance of the A balls was shortened to ·6 (tenths of inch). When the tinfoil spiral was used the critical spark distance at B rose to 6·4. When the iron wire bundle was inserted in the tube it did not in any perceptible degree increase this distance. The length of the sparking distance at A was 7·3, and when no alternative path was used at all to connect the jars the critical distance of the B balls, at which sparks sometimes passed and sometimes failed, was 11·1. Here, then, we have the non-magnetisability of iron by sudden discharges illustrated. Dr. Lodge has called attention to the fact that a "choking" coil having a core of divided iron and wound over with many turns of wire does not add to the apparent self-induction of a circuit discharging a Leyden jar. It may even diminish it when the discharge is oscillatory and of sufficient frequency, although the oscillations may be as few as 500 per second. This experiment shows, as we know from other facts, that eddy currents are set up even in a core of finely-divided iron, and that these eddy currents, under sufficiently rapid alternations, are confined to the surface of the core, and moreover, since they are as regards phase nearly in opposition to that of the current in the coil, they actually tend to diminish the total flux of induction through the coil, and hence diminish the self-induction of the circuit.

The inductive opposition to electric discharge presented by even a short length of conductor, when the difference of poten-

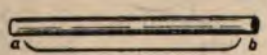

FIG. 17.

tial between the ends is made very suddenly, is seen in the tendency under such circumstances to *side flash*. If a conductor, say, a straight rod of copper, have one end to earth, and somewhere very near its side is the end of another conductor also "to earth," then if the free end of the first conductor is suddenly exalted in potential the impulsive rush of electricity meeting with such an obstacle in the inductance of the conductor spits or flashes out laterally and sparks to the other conductor. No conductor is able to prevent side flash altogether unless it has practically no inductance. As long as a conductor must be straight (like a lightning conductor) so long will there be a tendency to side flash. This is illustrated by the following experiment. A massive conductor has (Fig. 17) a very fine wire stretched alongside and air gaps in this byepath left by bringing the ends of the fine wire very near to the sides of the large conductor. On sending an impulsive rush of electricity through the large conductor little sparks are seen at *a* and *b*, showing that some of the discharge has left the thick conductor and travelled along the fine wire, even although it had to leap across an air gap. If the bare hands are applied to the ends of an open spiral of very stout copper wire, one end of which is connected to a "good earth," shocks will be felt when a Leyden jar is discharged through the copper. In this case the human body forms the bye-path, and the experiment indicates that the law of division of steady currents or slow discharges between conductors in parallel, viz., a division in the ratio of their conductivities, does not hold good for impulsive discharge, and that the relative inductances of the circuits have more influence in the latter case in determining what happens.

The distinction between the resulting discharges due to a steady electromotive force or strain and that due to an electromotive impulse or impulsive rush of electricity has been illustrated by some further experiments by Dr. Lodge on the behaviour of model lightning conductors when subjected to the action of these two modes of discharge. Two tin plates are placed horizontally and insulated, and these are supposed to represent the earth and a thunder-cloud. These plates are connected, as in Fig. 18, to an electrical machine, and by working the handle are brought up to a steady potential difference. On the lower plate are placed little rods of various heights, sharp, or having knobs, and these represent lightning conductors. At a certain potential difference the electric strain set up in the air exceeds the limit which the dielectric can sustain, and it breaks down, giving rise to a spark. A discharge then takes place towards one or other of the mimic lightning conductors. In one experiment three conductors were used—one with a large knob, ·9in. less in height than the distance

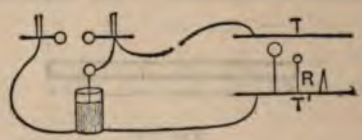

FIG. 18.

between the plates, the second with a small knob, 2in. less in height, and a sharp short point. The point even when very low prevents discharge altogether. It may be too low to be effective, or it may be insufficient to cope with the supply of electricity if that is supplied very fast, but it acts to prevent discharge. If the point is removed or covered up we then find that the discharge takes place, when the potential difference of the plates is made great enough, to the small knob by preference, and it does so even when the stem of the short knob is lower than that of the large knob. In other words, when the stems are the same height the small knob protects the large one, and it does this until lowered in height to about two inches less than the other; when this is the case both knobs are struck indifferently. And it does this even when a resistance of one megohm is interposed in the stem of the smaller

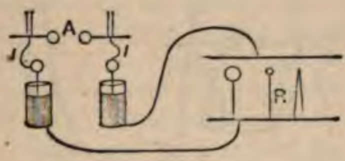

FIG. 19.

knob. The state of things is, however, very much altered if in place of bringing up the two plates gradually to a sparking potential difference they are very suddenly thrown into opposite electrical conditions by connecting them to the electrical machine, as shown in Fig. 19.

The jars charge up as they stand on the same wooden table, and when the potential rises to sparking amount they discharge at A, and a violent electric rush then takes place between the two plates, and the conductors between are struck. If the same three kind of conductors are used, and they be adjusted until they are all about equally struck, we find that the smaller and shorter-stemmed knob no longer protects the larger one, and the sharp point no longer protects either; all three, large ball, small ball and point, are liable to be struck equally if at the same height, and if they differ in height the highest is most likely to be struck, no matter what it is. Points are, then, no protection against these impulsive rushes of electricity. The special virtue of a point in the case of the slower-timed discharges is that it prepares the path of the discharge to itself, for in this case the path is pre-arranged by induction. If one of the conductors has a large resistance—say a liquid megohm inserted in it—then this one is no longer struck; it ceases to protect the other conductors even if higher than them, and even if it be so raised in height that it touches the top plate, thus connecting the plates by a bad conductor, the two other conductors get struck with apparently the same ease as before. This indicates that a lightning conductor with a bad earth cannot protect well against discharges of the nature of a sudden rush. Mr. Wimshurst has, however, shown reason for considering that in this experiment the electrical state of plates, as

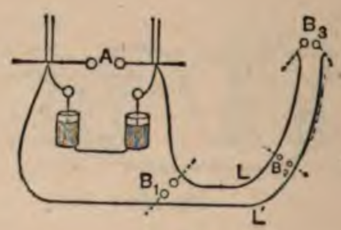

FIG. 20.

regards sign of electrification, may be of importance. The question how far the point protects from the impulsive rush is not altogether cleared up. It is still *sub judice.*

In performing the first experiment of the alternative path (Fig. 14) it was noticed that the B spark was longer than the A spark. Plainly this indicates that the discharge at A sets up electrical oscillations. The manner in which this is brought about is as follows:—On the commencement of the discharge the air-space is intensely heated, and its conductivity so far increased that the conditions as to the relation of inductance, resistance and capacity of the discharger and condenser are fulfilled, and the discharge takes the oscillatory form. If a couple of long leads are attached to the A discharger (Fig. 20), the farther ends being insulated, and a discharger B bridged across at B_1, B_2 or B_3, then it is found that at every discharge at A a spark can be obtained at B, and for a certain length of A spark the B spark will be longer at B_3 than at the nearer positions. Evidently what happens is that the electrical oscillation across the A discharge intervals sets up violent surgings to and fro in the open circuit wires, just like water in a long trough when it is tilted, and the recoil at the insulated ends, combined with the inductance of these leads, produces a cross flash at B. It is, in fact, a case of resonance; the long open circuit leads act like resonators to the oscillating discharge across A, and the nearer the length of the leads approaches to half a wave length or to some multiple of half a wave length the more perfect will be the resonance and the greater the recoil at the open ends, and hence the greater the spark at B_3.

If the experiment is tried in the dark, the B discharger being removed, it is seen that the leads glow at the ends with a vivid brush light at the moment when the jars are discharged. When the proper length of open circuit lead has been found which resonates best in accord with the jars used as dischargers, then the whole of the effects described can be made to disappear by connecting a very small Leyden jar to the ends of the wires. The increase of static capacity thus given to the leads reduces their potential below sparking point. Arranging the jar so as to leave an air-space between it and one of the wires, a spark passes into it at each A spark; but the jar is not in the least charged afterwards, proving that the spark is a double one, first in and then out of the jar, a real recoil of the reflected pulse. Hence, also, we see that the brush visible in the dark is the same on each wire, and one is not able to say that one brush is positive and the other negative, for each is both.

A curious experiment illustrating the electrical surgings or oscillations set up in a conductor which is suddenly discharged at one end is as follows:—Attach one end of a long wire to one knob of a Wimshurst machine, and connect the other pole to earth. The wire is otherwise insulated, and now forms one coating of a condenser of which the other is the walls of the room. The wire is bent round so that its free end nearly touches its initial end (*see* Fig. 21). Under these circumstances one would naturally say that a spark at B was absurd, and yet it is found that even if the wire is a stout copper wire a spark happens at B when one is produced at A. This B spark is caused by an electrical oscillation in the wire. The wire is, as it were, pumped full of electricity by the machine, and when the spark happens at A a release is given at that end for one brief instant. Then ensues a rebound of the electricity, and the pressure rises at the free end to sparking amount. The whole effect is just analogous to the effect of suddenly opening and closing a tap on a high-pressure water service—a concussion is heard in the tap on shutting, and if one could see the water it would be found that it rebounds, and a reflected wave is set up in the pipe, which, if the pipe is not strong enough, will burst it at some weak point. The practical moral of this is that any large conductor suddenly discharged has set up in it violent electrical surgings, which may cause it to spit

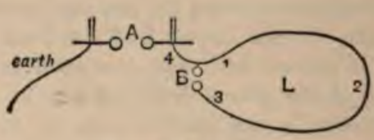

FIG. 21.

off discharges at other points, and these sparks may be as long as the principal spark.

Another way of making these electrical surgings conspicuous is by their effect in causing a Leyden jar to overflow, *i.e.*, to spark round its edge. A jar does this when its coatings are very suddenly raised to a great potential difference. Fig. 22 shows the arrangement. The inside of the jar is made to communicate direct to one machine pole and the outer coating through the intervention of a long wire to the other pole.

When a spark happens at A, and the length of the wire L is sufficiently great, the jar sparks over its edge. The explanation of this is as follows:—Whilst the handle of the machine is being turned the potential difference of the jar coatings increases. At a certain limit the air in the A space breaks down, and, being heated, becomes for a moment a very good conductor; there is, therefore, a rush of electricity out of the inner coating and into the outer coating, but the spark at A ceasing, this outflow from the jar is suddenly stopped and rebounds, whilst at the same time the inductance of the wire L causes a rush to continue into the jar. The rebound of the flow when the rush through the air space is suddenly stopped causes the potential difference of the coatings to rise to a point at which they spark over the edge of the glass. In an example given by Dr. Lodge the jar was a one gallon jar, with glass fully three inches above the tinfoil. L was a thick No. 1 copper wire circuit round a room. The jar overflows every time a spark happens at A, even though the length of this spark is only ·64in. If the long lead L is short-circuited, then the jar refuses to overflow until the A spark has been increased to 1·7in. The higher potential difference needed

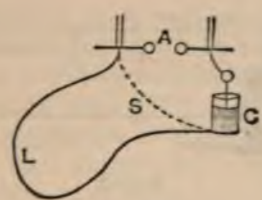

FIG. 22.

to cause overflow or rebound in the case with a short circuit is illustrative of the fact that a little self-induction in the discharging circuit bestows momentum on the flow and assists in making a back splash.

A hydraulic analogy to the above might be found in considering the case of a liquid flowing steadily along a trough or canal. If an obstruction was suddenly created, as by closing a valve or sluice, the liquid would rebound and a wave would be created; and, as in the case of the hydraulic ram, the rebound of the liquid against a closed valve might be made to lift some of it to a higher level than that from which it originally fell. In the electrical case, the rebound is made to raise the jar coatings to a greater potential difference than that which existed at the instant when the jar commenced to discharge.

§ 9. **Theory of Experiments on the Alternative Path.**—We may proceed, following Dr. Lodge,* to examine a little more in detail the electrical oscillations set up in an open circuit by Leyden jar discharges. These stationary electrical oscillations in linear conductors resemble those which can be set up in a cord fixed at one end, or in a trough of liquid, by suitably-timed impulses. As we have seen, if a jar discharges at A (*see* Fig. 23) in the ordinary way, simultaneously an even longer spark may be obtained at B, at the far end of two long open circuit leads. Or if the B ends of the wire are too far apart to allow of a spark, the wires glow and spit off brushes every time a discharge occurs at A. The theory of the effect seems to be that oscillations occur in the A circuit with a period $T = 2\pi\sqrt{LC}$, where L is the inductance of the A circuit and C the capacity of the jar. These oscillations disturb the

FIG. 23.

surrounding medium, and send out radiations of the precise nature of light, only too long in wave length to affect the retina of our eyes. The velocity of these electro-magnetic impulses is, as we have seen, equal to v, where

$$v = \frac{1}{\sqrt{\mu K}};$$

so the wave length of the oscillations is

$$\lambda = v T = 2\pi\sqrt{\frac{L}{\mu} \cdot \frac{C}{K}}.$$

Now $\frac{L}{\mu}$ is the electro-magnetic measure of inductance, and $\frac{C}{K}$ is the electrostatic measure of capacity, μ being the magnetic

* See *Phil. Mag.*, August, 1888; also *The Electrician*, August 10th, 1888, p. 435.

permeability, and K the electrostatic inductivity of the medium surrounding the wire.

Each of these quantities is of the dimensions of a *length*, and the wave length of the radiation is 2π times their geometric mean. We may look upon it, then, that the magnetic field due to the oscillatory current in the A circuit, which circuit consists partly of metal wires, partly of the dielectric of the jar, and partly of the heated air in the spark space, acts inductively upon the other or B circuit which is adjacent to it, and has, in fact, the jar dielectric as a common boundary. The pulsating field induces oscillatory currents in the open B circuit. These electric pulses rush along the surface of the wires with a certain amount of dissipation, and are reflected at the distant end, producing a recoil kick or impulse tending to break down the dielectric in the air gap B with production of a spark. These currents continue to oscillate to and fro until damped out of existence by the resistance of the wires. The best effect in the way of spark at B is observed when the length of each wire is such that the time occupied by an electric pulse in travelling along the wires and back again is equal to the time of a complete oscillation in the A circuit; that is, when the length of the open circuit wires is equal to half a wave length or to some multiple of half a wave length. The natural period of oscillation in the long wires will then agree with the oscillation period of the discharging circuit and the oscillations in the open circuit wires, and the field due to the oscillations in the A circuit will vibrate in unison like a column of air in a pipe resonating a tuning fork, or like a string vibrating when attached to the tongue of a reed.

The elementary theory of the open circuit oscillations is as follows:—

Let l_1 and r_1 be the inductance and resistance of the straight wires per unit of length, *as affected by the periodicity*, and let c_1 be the capacity per unit of length. Now it has been shown by Lord Rayleigh (*Phil. Mag.*, May, 1886) that with very rapid oscillations owing to the circumferential distribution of the current the inductance and resistance have values different from the steady current values, and when the frequency of the oscillations is very great the resistance r_1 per unit of length is the geometric mean of its ordinary value r and $\frac{1}{2} p \mu_0$, where μ_0 is the magnetic permeability of the material of the conductor, or $r_1 = \sqrt{\frac{1}{2} p \mu_0 r}$, p being, as usual, $2\pi n$, n being the number of complete oscillations per second.

And again, when n is very great, the inductance L_1 per unit of length is equal to a constant *plus* $\frac{r}{p}$, or

$$l_1 = l + \frac{r_1}{p},$$

l being the induction for slowly fluctuating currents.

In the case of the two parallel wires we have for the slope of the potential $-\frac{dV}{dx}$ along them the usual equations,

$$l_1 \frac{di}{dt} + r_1 i = -\frac{dV}{dn} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

i being the instantaneous current in the section of the length lying at a distance x from the origin; and also for the accumulation of charge in this element dx of the length we have the equation

$$-\frac{dV}{dt} = \frac{1}{C_1} \frac{di}{dx} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The elimination of i between these equations gives us a differential equation for V, and shows that stationary waves of current are set up in finite wires of suitable length supplied with alternating electromotive force. The solution of the equation for a long wire when r_1 is small and p is very big is

$$V = V_0 e^{-\frac{m_1 x}{n_1}} \cos p \left(t - \frac{x}{n_1}\right),$$

where $$m_1 = \frac{r_1}{2 l_1} \text{ and } n_1 = \frac{1}{\sqrt{L_1 C_1}}.$$

The velocity of propagation of the wave is therefore n_1 and th[e] wave length is $\frac{2\pi}{p} n_1$.

For two parallel wires, as in the Leyden jar case, we have f[or] each wire

$$r_1 = \sqrt{\tfrac{1}{2} p \mu_0 r},$$

r being the ordinary resistance. And again, as Lord Rayleigh has shown (*Phil. Mag.*, May, 1886), we have

$$l_1 = 4\,\mu \log \frac{b}{a} + \frac{r_1}{p},$$

b being the distance between the parallel wires and a the radius of either, and μ the magnetic permeability of the material of the conductors.

For immensely quick oscillations the second term is zero. Also, the capacity C_1 of the wires per unit of length is, by a known theorem,

$$C_1 = \frac{K}{4 \log \frac{b}{a}};$$

hence

$$\frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{\mu K}},$$

and the velocity of the pulse along the wires is the same as in the dielectric round them. In other words, the electric pulses set up in the wires rush to and fro with a velocity equal to that with which the electro-magnetic impulse is propagated through the dielectric round them. Hence, we have here a means of determining experimentally the wave length of a given discharging circuit. Either vary the size of the A circuit or adjust the length of the B wires until the recoil spark B is as long as possible. Then measure, and see whether the length of each wire is not equal to

$$\pi \sqrt{\frac{L}{\mu} \cdot \frac{C}{\kappa}}.$$

A small condenser can be made having an electrostatic capacity of, say, two or three centimetres, and if such a coated pane be made to discharge over its edge, the discharge circuit will have an electro-magnetic inductance of a few centimetres. Under these circumstances the electrical oscillations would be at the rate of a thousand million a second, and the wave length of the electro-magnetic disturbance radiated would be about 20 to 30 centimetres.

If a conductor as small as an atom could have its electrical charge disturbed in the same way, oscillations would be set up of the frequency of light waves and electro-magnetic disturbances of light wave length radiated; and it seems probable that this is just what light waves are, viz., electro-magnetic disturbances propagated through the ether and due to electric oscillations set up in the atomic charge.

§ **10. Impulsive Impedance.**—In the experiments of the "alternative path," as described by Dr. Lodge, the main result is very briefly summed up by saying that when a sudden discharge had to pass through a conductor it was found that iron and copper acted about equally well, and indeed iron sometimes exhibited a little superiority, and that the thickness of the conductor and its ordinary conductivity mattered very little indeed. We are led by this to see that the impedance which a conductor offers to a sudden discharge, and which may be called its impulsive impedance, is something quite different from its ordinary or ohmic resistance, or even its impedance, defined as $\sqrt{R^2+p^2L^2}$, to slowly periodic or oscillatory currents. As already mentioned, the resistance of a conductor to very rapidly changing currents is expressed by R_1, where

$$R_1 = \sqrt{\tfrac{1}{2} p\, l\, \mu_0 R},$$

R being the resistance to steady currents, μ_0 the permeability of the material of the conductor and l its length, and $p = 2\pi$ times the frequency of the oscillation. Also the corresponding inductance L_1 is

$$L_1 = L + \frac{R_1}{p},$$

where L is a constant depending on the size and form of the circuit, but only in a small degree upon its thickness. Hence, forming the function $\sqrt{R_1^2+p^2L_1^2}$, and calling this Im_1, we have

$$Im_1 = \sqrt{(pL+R_1)^2+R_1^2}$$

$$= pL\sqrt{1+\frac{2m}{\sqrt{p}}+\frac{2m^2}{p}},$$

where $$m = \frac{1}{L}\sqrt{\tfrac{1}{2} l\, \mu_0 R}.$$

In the case of enormously rapid oscillations the value of Im_1 practically reduces to p L, and hence the impulsive impedance varies in simple proportion to the frequency, and depends on the form and size of the circuit, but not at all on its specific resistance, magnetic permeability, or diameter.

All this is borne out by experiment. In some of his experiments Dr. Lodge found the impedance of a No. 2 wire of two and a-half metres length bent into a circle to be 180 ohms at twelve million oscillations per second, and for a No. 40 wire the impedance was only 300 ohms, although the ohmic resistances of these wires were respectively ·004 ohms and 2·6 ohms. At three million oscillations per second, or at one-fourth the frequency, the impedances of the same circuits were 43 ohms and 78 ohms. At one-quarter million oscillations per second the impedances are reduced to four and six ohms respectively for the thick rod and the fine wire. Hence, for frequences of a million per second and upwards, such as occur in jar discharges, and perhaps in lightning, the impedance of all reasonably conducting circuits is the same, and independent of conductivity and permeability, and hardly affected greatly by enormous changes in diameter.

§ 11. Dr. Hertz's Researches on Electrical Oscillatory Induction.—The whole question of electrical oscillatory discharge has been made the subject of some elaborate and most instructive investigations by Dr. Hertz, of Carlsruhe. As these results have a most direct bearing on the physical theory of currents of induction, we shall reproduce here an abstract of these researches, which is from the pen of Mr. G. W. de Tunzelmann.*

Preliminary Experiments.—It is known that if in the secondary circuit of an induction coil there be inserted, in addition to the ordinary air space across which sparks pass, a Riess spark micrometer, with its poles joined by a long wire, the discharge will pass across the air space of the micrometer in preference to following the path of least resistance through the wire, provided

* This section originally appeared as a series of articles in the pages of *The Electrician*, in Vol. XXI., pp. 587, 625, 663, 696, 725, 757, 788 (1888). The writer felt it would be difficult to make a more complete digest of Dr. Hertz's work than is contained in these admirable articles, and by kind permission of their author is allowed to reproduce them in these pages.

this air space does not exceed a certain limit; and it is upon this principle that lightning protectors for telegraph lines are constructed. It might be expected that the sparks could be made to disappear by diminishing the length and resistance of the connecting wire; but Hertz finds that though the length of the sparks can be diminished in this way, it is almost impossible to get rid of them entirely, and they can still be observed when the balls of the micrometer are connected by a thick copper wire only a few centimetres in length.

This shows that there must be variations in the potential measurable in hundredths of a volt in a portion of the circuit

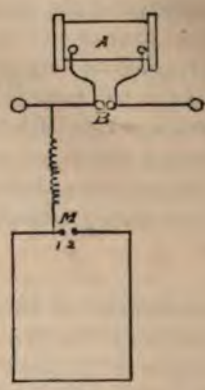

Fig. 24.

only a few centimetres in length, and it also gives an indirect proof of the enormous rapidity of the discharge, for the difference of potential between the micrometer knobs can only be due to self-induction in the connecting wire. Now the time occupied by variations in the potential of one of the knobs must be of the same order as that in which these variations can be transmitted through a short length of a good conductor to the second knob. The resistance of the wire connecting the knobs is found to be without sensible effect on the results.

In Fig. 24, A is an induction coil and B a discharger. The wire connecting the knobs 1 and 2 of the spark micrometer M consists of a rectangle, half a metre in length, of copper wire two millimetres in diameter. This rectangle is connected with the secondary circuit of the coil in the manner shown in the diagram, and, when the coil is in action, sparks, sometimes several millimetres in length, are seen to pass between the knobs 1 and 2, showing that there are violent electrical oscillations not only in the secondary circuit itself, but in any conductor in contact with it. This experiment shows even more clearly than the previous one that the rapidity of the oscillations is comparable with the velocity of transmission of electrical disturbances through the copper wire, which, according to all the evidence at our disposal, is nearly equal to the velocity of light.

In order to obtain micrometer sparks some millimetres in length a powerful induction coil is required, and the one used by Hertz was 52 centimetres in length and 20 centimetres in diameter, provided with a mercury contact-breaker, and excited by six large Bunsen cells. The discharger terminals consisted of brass knobs three centimetres in diameter. The experiments showed that the phenomenon depends to a very great extent on the nature of the sparks at the discharger, the micrometer sparks being found to be much weaker when the discharge in the secondary circuit took place between two points or between a point and a plate than when knobs were used. The micrometer sparks were also found to be greatly enfeebled when the secondary discharge took place in a rarefied gas, and also when the sparks in the secondary were less than half a centimetre in length; while, on the other hand, if they exceeded $1\frac{1}{2}$ centimetre the sparks could no longer be observed between the micrometer knobs. The length of secondary spark which was found to give the best results, and which was therefore employed in the further observations, was about three-quarters of a centimetre.

Very slight differences in the nature of the secondary sparks were found to have great effect on those of the micrometer, and Hertz states that after some practice he was able to determine at once from the sound and appearance of the secondary spark whether it was of a kind to give the most powerful effects at the micrometer. The sparks which gave the best results were of a brilliant white colour, only slightly jagged, and accompanied by a sharp crack.

The influence of the spark is readily shown by increasing the distance between the discharger knobs beyond the striking distance, when the micrometer sparks disappear entirely, although the variations of potential are now greater than before. The length of the micrometer circuit has naturally an important influence on the length of the spark, as the greater its length the greater will be the retardation of the electrical wave in its passage through it from one knob of the micrometer to the other.

The material, the resistance, and the diameter of the wire of which the micrometer circuit is formed have very little influence on the spark. The potential variations cannot, therefore, be due to the resistance; and this was to be expected, for the rate of propagation of an electrical disturbance along a conductor depends mainly on its capacity and coefficient of self-induction, and only to a very small extent on its resistance. The length of the wire connecting the micrometer circuit with the secondary circuit of the coil is also found to have very little influence, provided it does not exceed a few metres in length. The electrical disturbances must therefore traverse it without undergoing any appreciable change. The position of the point of the micrometer circuit which is joined to the secondary circuit is, on the other hand, of the greatest importance, as would be expected, for, if the point is placed symmetrically with respect to the two micrometer knobs, the variations of potential will reach the latter in the same phase, and there will be no effect, as is verified by observation. If the two branches of the micrometer circuit on each side of the point of contact of the connection with the secondary are not symmetrical the spark cannot be made to disappear entirely; but a minimum effect is obtained when the point of contact is about half-way between the micrometer knobs. This point may be called the null point.

Fig. 25 shows the arrangement employed, *e* being the null point of the rectangular circuit, which is 125 centimetres long by 80 centimetres broad. When the point of contact is at *a* or *b* sparks of from three to four millimetres in length are observed; when it is at *e* no sparks are seen, but they can be made to reappear by shifting the point of contact a few centimetres to the right or left of the null point. It should be noted that sparks only a few hundredths of a millimetre in length can be observed. If, when the point of contact is at *e*, another conductor is placed in contact with one of the micrometer knobs, the sparks reappear.

Now, the addition of this conductor cannot produce any alteration in the time taken by the disturbances proceeding from *e* to reach the knobs, and therefore the phenomenon cannot be due simply to single waves in the directions *c a* and *d b* respectively, but must be due to repeated reflection of the waves until a condition of stationary vibration is attained, and the addition of the conductor to one of the knobs must

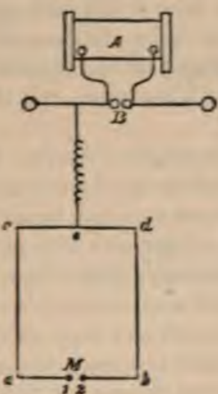

FIG. 25.

diminish or prevent the reflection of the waves from that terminal. It must be assumed, then, that definite oscillations are set up in the micrometer circuit just as an elastic bar is thrown into definite vibrations by blows from a hammer. If this assumption is correct, the condition for the disappearance of the sparks at M will be that the vibration periods of the two branches *e* 1 and *e* 2 shall be equal. These periods are determined by the products of the coefficients of self-induction of these conductors into the capacities of their terminals, and are practically independent of their resistances.

In confirmation of this it is found that if, when the point of contact is at *e* and the sparks have been made to reappear by connecting a conductor with one of the knobs, this conductor is replaced by one of greater capacity, the sparking is greatly increased. If a conductor of equal capacity is connected with the other micrometer knob, the sparks disappear again; the effect of the first conductor can also be counteracted by shifting the point of contact towards it, thereby diminishing the self-induction in that branch. The conclusions were further confirmed by the results obtained when coils of copper wire were inserted into one or other and then into both of the branches of the micrometer circuit.

Hertz supposed that as the self-induction of iron wires is, for slow alternations, from eight to ten times that of copper wires, therefore a short iron wire would balance a long copper one; but this was not found to be the case, and he concludes that, owing to the great rapidity of the alternations, the magnetism of the iron is unable to follow them, and therefore has no effect on the self-induction.*

Induction Phenomena in Open Circuits.—In order to test more fully his conclusion that the sparks obtained in the experiments described in my last Paper were due to self-induction, Dr. Hertz placed a rectangle of copper wire with sides 10 and 20 centimetres in length respectively, broken by a short air space, with one of its sides parallel and close to various portions of the secondary circuit of the coil and of the micrometer circuit, with solid dielectrics interposed to obviate the possibility of sparking across, and he found that sparking in this rectangle invariably accompanied the discharges of the induction coil, the longest sparks being obtained when a side of the rectangle was close to the discharger.

A copper wire, *i g h* (Fig. 26), was next attached to the discharger, and a side of the micrometer circuit, which was supported on an insulating stand, was placed parallel to a portion of this wire, as shown in the diagram. The sparks at M were then found to be extremely feeble until a conductor, C,

* In a note in Wiedemann's *Annalen*, Vol. XXXI., p. 543, Dr. Hertz states that since the publication of his Paper in the same volume he had found that von Bezold had published a Paper, in 1870 (Poggendorff's *Annalen*, Vol. CXL., p. 541), in which he had arrived by a different method of experimenting at similar results and conclusions as those given by him under the head of Preliminary Experiments.

was attached to the free end, *h*, of the copper wire, when they increased to one or two millimetres in length. That the action of C was not an electrostatic one was shown by its producing no effect when attached at *g* instead of at *h*. When the knobs of the discharger B were so far separated that no sparking took place there, the sparks at M were also found to disappear, showing that these were due to the sudden discharges and not to the charging current. The sparks at the discharger which produced the most effect at the micrometer were of the same

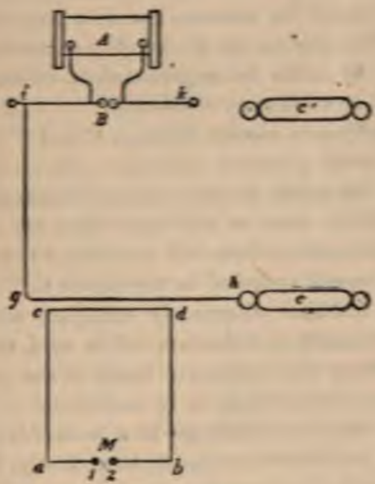

FIG. 26.

character as those described in my last Paper. Sparks were also found to occur between the micrometer circuit and insulated conductors in its vicinity. The sparks became much shorter when conductors of larger capacity were attached to the micrometer knobs, or when these were touched by the hand, showing that the quantity of electricity in motion was too small to charge these conductors to a similarly high potential. Joining the micrometer knobs by a wet thread did not perceptibly diminish the strength of the sparks. The effects in the micrometer circuit were not of sufficient strength to produce any sensation when it was touched or the circuit completed through the body.

In order to obtain further confirmation of the oscillatory nature of the current in the circuit *k i h g* (Fig. 26), the conductor C was again attached to *h*, and the micrometer knobs drawn apart until sparks only passed singly. A second conductor, C′, as nearly as possible similar to C, was then attached to *k*, when a stream of sparks was immediately observed, and it continued when the knobs were drawn still further apart. This effect could not be ascribed to a direct action of the portion of circuit *i k*, for in this case the action of the portion of circuit *g h* would be weakened, and it must therefore have consisted in C′ acting on the discharging current of C—a result which would be quite incomprehensible unless the current in *g h* were of an oscillatory character.

Since an oscillatory motion between C and C′ is essential for the production of powerful inductive effects, it will not be sufficient for the spark to occur in an exceedingly short time, but the resistance must at the same time not exceed certain limits. The inductive effects will therefore be excessively small if the induction coil included in the circuit C C′ is replaced by an electrical machine alternately charging and discharging itself, or if too small an induction coil is used, or, again, if the air space between the discharger knobs is too great, as in all these cases the motion ceases to be oscillatory.

The reason that the discharge of a powerful induction coil gives rise to oscillatory motion is that, firstly, it charges the terminals C and C′ to a high potential; secondly, it produces a sudden spark in the intervening circuit; and thirdly, as soon as the discharge begins the resistance of the air space is so much reduced as to allow of oscillatory motion being set up. If the terminal conductors are of a very large capacity—for example, if the terminals are in connection with a battery—the current of discharge may indefinitely reduce the resistance of the air space, but when the terminal conductors are of small capacity this must be done by a separate discharge, and therefore, under the conditions of the author's experiments, an induction coil was absolutely essential for the production of the oscillations.

As the induced sparks in the experiment last described were several millimetres in length, the author modified it by using the arrangement shown in Fig. 27, and greatly increasing the distance between the micrometer circuit and the secondary circuit of the induction coil. The terminal conductors C and C′ were three metres apart, and the wire between them was of copper, 2 millimetres in diameter, with the discharger B at its centre.

The micrometer circuit consisted, as in the preceding experiments, of a rectangle 80 centimetres broad by 120 centimetres long. With the nearest side of the micrometer circuit at a distance of half a millimetre from C B C′, sparks two millimetres in length were obtained at M, and though the length of the sparks decreased rapidly as the distance of the micrometer circuit was increased, a continuous stream of sparks was still obtained at a distance of one and a-half metre. The intervention of the observer's body between the micrometer circuit and the wire C B C′ produced no visible effect on the stream of sparks at M. That the effect was really due to the rectilinear conductor C B C′ was proved by the fact that when one or other, or both, halves of this conductor were removed, the sparks at M ceased. The same effect was produced by drawing the knobs of the discharger B apart until sparks ceased to pass, showing

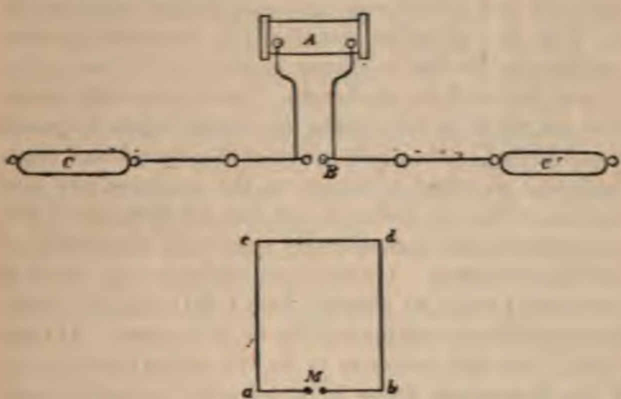

FIG. 27.

that the effect was not due to the electrostatic potential difference of C and C′, as this would be increased by separating the discharger knobs beyond sparking distance.

The closed micrometer circuit was then replaced by a straight copper wire, slightly shorter than the distance C C′, placed parallel to C B C′ and at a distance of 60 centimetres from it. This wire terminated in knobs, 10 centimetres in diameter, attached to insulating supports, and the spark micrometer divided it into two equal parts. Under these circumstances sparks were obtained at the micrometer as before.

With the rectilinear open micrometer circuit sparks were still observed at the micrometer when the discharger knobs of the secondary coil circuit were separated beyond sparking distance. This was, of course, due simply to electrostatic induction, and shows that the oscillatory current in C C′ was superposed upon the ordinary discharges. The electrostatic action could be got rid of by joining the micrometer knobs by means of a damp thread. The conductivity of this thread was therefore sufficient to afford a passage to the comparatively slow alternations of the coil discharge, but was not sufficient to provide a passage for the immeasurably more rapid alternations of the oscillatory current. Considerable sparking took place at the micrometer when its distance from C B C′ was 1·2 metre, and faint sparks were distinguishable up to 3 metres. At these distances it was not necessary to use the damp thread to get rid of the electrostatic action, as, owing to its diminishing more rapidly with increase of distance than the effect of the current induction, it was no longer able to produce sparks in the micrometer, as was proved by separating the discharger knobs beyond sparking distance, when sparks could no longer be perceived at the micrometer.

Resonance Phenomena.—In order to determine whether, as some minor phenomena had led the author to suppose, the oscillations were of the nature of a regular vibration, he availed himself of the principle of resonance. According to this principle, an oscillatory current of definite period would, other conditions being the same, exert a much greater inductive effect upon one of equal period than upon one differing even slightly from it.*

* *See* Oberbeck, Wiedemann's *Annalen*, Vol. XXVI., p. 245, 1885.

If, then, two circuits are taken having as nearly as possible equal vibration periods, the effect of one upon the other will be diminished by altering either the capacity or the coefficient of self-induction of one of them, as a change in either of them would alter the period of vibration of the circuit.

This was carried out by means of an arrangement very similar to that of Fig. 27. The conductor C C′ was replaced by a straight copper wire 2·6 metres in length and 5 millimetres in diameter, divided into two equal parts as before by a discharger. The discharger knobs were attached directly to the secondary terminals of the induction coil. Two hollow zinc spheres, 30 centimetres in diameter, were made to slide on the wire, one on each side of the discharger, and since, electrically speaking, these formed the terminals of the conductor, its length could be varied by altering their position. The micrometer circuit was chosen of such dimensions as to have, if the author's hypothesis were correct, a slightly shorter vibration period than that of C C′. It was formed of a square, with sides 75 centimetres in length, of copper wire 2 millimetres in diameter, and it was placed with its nearest side parallel to C B C′ and at a distance of 30 centimetres from it. The sparking distance at the micrometer was then found to be 0·9 millimetre. When the terminals of the micrometer circuit were placed in contact with two metal spheres 8 centimetres in diameter, supported on insulating stands, the sparking distance could be increased up to 2·5 millimetres. When these were replaced by much larger spheres the sparking distance was diminished to a small fraction of a millimetre. Similar results were obtained on connecting the micrometer terminals with the plates of a Kohlrausch condenser. When the plates were far apart the increase of capacity increased the sparking distance, but when the plates were brought close together the sparking distances again fell to a very small value.

The simplest method of adjusting the capacity of the micrometer circuit is to suspend to its ends two parallel wires the distance and lengths of which are capable of variation. By this means the author succeeded in increasing the sparking distance up to three millimetres, after which it diminished when the wires were either lengthened or shortened. The decrease of the sparking distance on increasing the capacity was naturally to be expected; but it would be difficult to understand, except on the principle of resonance, why a decrease of the capacity should have the same effect.

The experiments were then varied by diminishing the capacity of the circuit C B C′ so as to shorten its period of oscillation, and the results confirmed those previously obtained; and a series of experiments in which the lengths and capacities of the circuits were varied in different ways showed conclusively that the maximum effect does not depend on the conditions of either one of the two circuits, but on the existence of the proper relation between them.

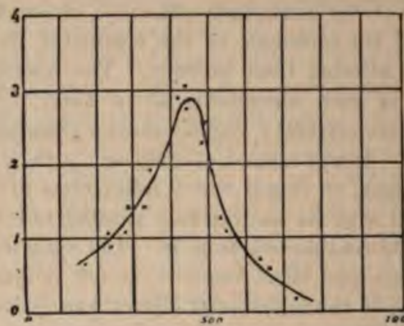

Fig. 28.

Curve showing relation between length of side of rectangle (taken as abscissa) and maximum sparking distance (taken as ordinate), the sides consisting of straight wires of varying lengths.

When the two circuits were brought very close together, and the discharger knobs separated by an interval of 7 millimetres, sparks were obtained at the micrometer, which were also 7 millimetres in length, when the two circuits had been carefully adjusted to have the same period. The induced E.M.F.s must in this case have attained nearly as high a value as the inducing ones.

To show the effect of varying the coefficient of self-induction, a series of rectangles, *a b c d* (Fig. 27), were taken, having a constant breadth, *a b*, but a length, *a c*, continually increasing from 10 centimetres up to 250 centimetres: it was found that the maximum effect was obtained with a length of 1·8 metre.

The quantitative results of these experiments are shown in Fig. 28, in which the abscissæ of the curve are the double lengths of the rectangles, and the ordinates represent the corresponding maximum sparking distances. The sparking distances could not be determined with great exactness, but the errors were not sufficient to mask the general nature of the result.

In a second series of experiments the sides *a c* and *b d* were formed of loose coils of wire which were gradually pulled out, and the result is shown in Fig. 29. It will be seen that the maximum sparking distance was attained for a somewhat

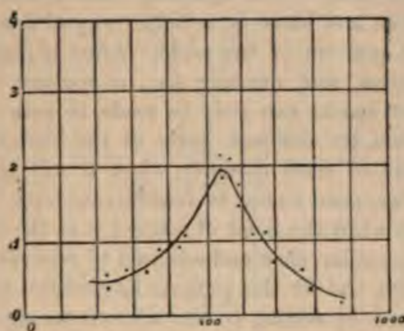

FIG. 29.

Curve showing relation between length of side of rectangle (taken as abscissa) and maximum sparking distance (taken as ordinate), the sides consisting of spirals gradually drawn out.

greater length of side, which is explained by the fact that in the latter experiments the self-induction only was increased by increase of length, while in the former series the capacity was increased as well. Varying the resistance of the micrometer circuit by using copper and German silver wires of various diameters was found to have no effect on the period of oscillation, and extremely little on the sparking distance.

When the wire *c d* was surrounded by an iron tube, or when it was replaced by an iron wire, no perceptible effect was obtained, confirming the conclusion previously arrived at that the magnetism of the iron is unable to follow such rapid oscillations, and therefore exerts no appreciable effect.

Nodes.—The vibrations in the micrometer circuit which have been considered are the simplest ones possible, but not the only ones. While the potential at the ends alternates between two fixed limits, that at the central portion of the circuit retains a constant mean value. The electrical vibration, therefore, has a node at the centre, and this will be the only nodal point. Its existence may be proved by placing a small insulated sphere close to various portions of the micrometer circuit while sparks are passing at the discharger of the coil, when it will be found that if the sphere is placed close to the centre of the circuit the sparking will be very slight, increasing as the sphere is moved further away. The sparking cannot, however, be entirely got rid of, and there is a better way of determining the existence and position of the node. After adjusting the two circuits to unison, and drawing the micrometer terminals so far apart that sparks can only be made to pass by means of resonant action, let different parts of the circuit be touched by a conductor of some capacity, when it will be found that the sparks disappear, owing to interference with the resonant action, except when the point of contact is at the centre of the circuit. The author then endeavoured to produce a vibration with two nodes, and for this purpose he modified the apparatus previously used by adding to the micrometer circuit a second rectangle, *efgh*, exactly similar to the first (as shown in Fig. 30), and joining the points of the circuit near the terminals by wires 1 3 and 2 4, as shown in the diagram.

The whole system then formed a closed metallic circuit, the fundamental vibration of which would have two nodes. Since the period of this vibration would necessarily agree closely with that of each half of the circuit, and, therefore, with that of the circuit C C′, it was to be expected that the vibration would have a pair of loops at the junctions 1 3 and 2 4, and a pair of nodes at the middle points of *c d* and *g h*. The vibrations were determined by measuring the sparking distance between the micrometer terminals 1 and 2. It was found that, contrary to what was expected, the addition of the second rectangle diminished this sparking distance from about three millimetres to about one millimetre. The existence of resonant action between the circuit C C′ and the micrometer circuit was, however, fully demonstated, for any alteration in the circuit *efgh*, whether it consisted in increasing or in decreasing its length, diminished the sparking distance. It was also found that much weaker sparking took place between *cd* or *gh* and an insulated sphere than between *ae* or *bf* and the same sphere, showing that the nodes were in *cd* and *gh*, as expected. Further, when the sphere was made to touch *cd* or *gh* it had no effect on the sparking distance of 1 and 2; but when the point of contact was at any other portion of the circuit the sparking distance was diminished, showing that these nodes did really belong to the vibration, the resonant action of which increased this sparking distance.

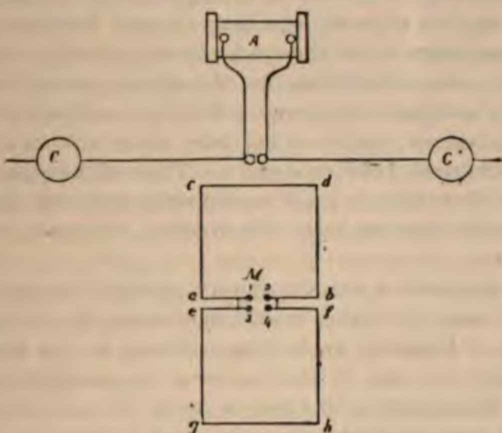

FIG. 30.

The wire joining the points 2 and 4 was then removed. As the strength of the induced oscillatory current should be zero at these points, the removal ought not to disturb the vibrations, and this was shown experimentally to be the case, the resonant effects and the position of the nodes remaining unchanged. The vibration with two nodal points was, of course, not the fundamental vibration of the circuit, which consisted of a vibration with a node between *a* and *e*, and for which the highest values of the potential were at the points 2 and 4.

When the spheres forming the terminals at these points were brought close together slight sparking was found to take place between them, which was attributed to the excitation, though only to a small extent, of the fundamental vibration. This explanation was confirmed in the following manner:—The sparks between 1 and 2 were broken off, leaving only the sparks between 2 and 4, which measured the intensity of the fundamental vibration. The period of vibration of the circuit C C′ was then increased by drawing it out to its full length, and thereby increasing its capacity, when it was observed that the sparking gradually increased to a maximum, and then began to diminish again. The maximum value must evidently occur when the period of vibration of the circuit C C′ is the same as that of the fundamental vibration of the micrometer circuit, and it was shown that when the sparking distance between 2 and 4 had its maximum value the sparks corresponded to a vibration with only one nodal point, for the sparks ceased when the previously existing nodes were touched by a conductor, and the only point where contact could take place without effect on the sparking was between *a* and *e*. These results show that it is possible to excite at will in the same conductor either the fundamental vibration or its first overtone, to use the language of acoustics.

Hertz appears to consider it very doubtful whether it will be possible to get higher overtones of electrical vibration, the difficulty of obtaining such lying not only in the method of observation, but also in the nature of the oscillations themselves. The intensity of these is found to vary considerably during a series of discharges from the coil even when all the circumstances are maintained as constant as possible, and the comparative feebleness of the resonant effects shows that there must be a considerable amount of damping. There are, moreover, many secondary phenomena which seem to indicate that irregular vibrations are superposed upon the regular ones, as would be expected in complex systems of conductors. If, therefore, we wish to compare electrical oscillations from a mathematical point of view with those of acoustics, we must seek our analogy in the high notes intermixed with irregular vibrations, obtained, say, by striking a wooden rod with a hammer rather than in the comparatively slow harmonic vibration of tuning forks or strings; and in the case of vibrations of the former class we have to be contented even in the study of acoustics with little more than indications of such phenomena as resonance and nodal points.

Referring to the conditions to be fulfilled in order to obtain the best results, should other physicists desire to repeat the experiments, Dr. Hertz notes a fact of very considerable interest and novelty, namely, that the spark from the discharger should always be visible from the micrometer, as, when this was not the case, though the phenomena observed were of the same character the sparking distance was invariably diminished. This effect of the light from the spark of an induction coil in increasing the sparking distance in the secondary circuit of another coil excited great interest when referred to by Prof. Lodge in the course of the recent discussion on Lightning Conductors at the British Association, and he pointed out that the same effect was produced by light from burning magnesium wire or other sources rich in the ultra-violet rays.

Theory of the Experiments.—The theories of electrical oscillations which have been developed by Sir William Thomson, von Helmholtz, and Kirchhoff have been shown* to hold good for the open circuit oscillations of induction apparatus, as well as for the oscillatory Leyden jar discharge; and although Dr. Hertz has not succeeded in obtaining definite quantitative results to compare with theory, it is of interest to inquire whether the observed results are of the same order as those indicated by theory.

Hertz considers, in the first place, the vibration period. Let T be the period of a single or half vibration proper to the conductor exciting the micrometer circuit; L its coefficient of self-induction in absolute electro-magnetic measure, expressed, therefore, in centimetres; C the capacity of one of its terminals in electrostatic measure, and therefore also expressed in centimetres; and v the velocity of light in centimetre-seconds; then, if the resistance of the conductor is small,

$$T = \frac{\pi\sqrt{LC}}{v}.$$

In the case of the resonance experiments, the capacity C was approximately the radius of the sphere forming the terminal,

* Lorentz, Wiedemann's *Annalen*, Vol. VII., p. 161, 1879.

so that C = 15 centimetres. The coefficient of self-induction was that of a wire of length $l = 150$ centimetres and diameter $d = 1/2$ centimetre.

According to Neumann's formula,

$$L = \iint \frac{\cos. \epsilon}{r} \, d s \, d s',$$

which gives in the case considered

$$L = 2 l \left(\log. \frac{4 l}{d} - 0{\cdot}75 \right) = 1{,}902 \text{ centimetres.}$$

As, however, it is not quite certain that Neumann's formula is applicable to an open circuit, it is better to use von Helmholtz's more general formula, containing an undetermined constant k, according to which

$$L = 2 l \left(\log. \frac{4 l}{d} - 0{\cdot}75 + \frac{1-k}{2} \right).$$

Putting $k = 1$, this reduces to Neumann's formula; for $k = 0$ it reduces to that of Maxwell, and for $k = -1$ to Weber's. The greatest difference in the values of L obtained by giving these different values to k would not exceed a sixth of its mean value, and therefore, for the purposes of the present approximation it is enough to assume that k is not a large positive or negative number; for if the number 1,902 does not give the correct value of the coefficient for the wire 150 centimetres in length, it will give the value corresponding to a conductor not differing greatly from it in length.

Taking L = 1,902 centimetres, we have $\pi \sqrt{C L} = 531$ centimetres, which represents the distance traversed by light during the oscillation, or, according to Maxwell's theory, the length of an electro-magnetic ether wave. The value of T is then found to be 1·77 hundred-millionths of a second, which is of the same order as the observed results.

The ratio of damping is then considered. In order that oscillations may be possible, the resistance of the open circuit must be less than $2 v \sqrt{L/C}$. For the exciting circuit used this gives 676 ohms as the upper limit of resistance. If the actual resistance, r, is sensibly below this limit, the ratio of damping will be $e^{\frac{rT}{2L}}$. The amplitude will therefore be reduced in the ratio 1:2·71 in

$$\frac{2L}{rT}=\frac{2v}{\pi r}\sqrt{\frac{L}{C}}=\frac{676}{\pi r}=\frac{215}{r}$$

oscillations. We have, unfortunately, no means of determining the resistance of the air space traversed by the spark, but as the resistance of a strong electric arc is never less than a few ohms we shall be justified in assuming this as the minimum limit. From this it would follow that the number of oscillations due to a single impulse must be reckoned in tens, and not in hundreds or thousands, which is in accordance with the character of the experimental results and agrees with results observed in the case of the oscillatory Leyden jar discharge. In the case of closed metallic circuits, on the other hand, theory indicates that the number of oscillations before equilibrium is attained must be reckoned by thousands.

Hertz compares, lastly, the order of the inductive actions of these oscillations according to theory with that of the effects actually observed. To do this it must be noted that the maximum E.M.F. induced by the oscillation in its own circuit is approximately equal to the maximum potential difference at its extremities; for if there were no damping these quantities would be identical, since at any moment the potential difference at the extremities and the E.M.F. of induction would be in equilibrium. In the experiments under consideration the potential difference at the extremities was such as to give a spark 7 to 8 millimetres in length, which must therefore represent the maximum inductive action excited in its own circuit by the oscillation. Again, at any instant the induced E.M.F. in the micrometer circuit must be to that in the exciting conductor in the same ratio as that of the coefficient of mutual induction p of the two circuits to the coefficient of self-induction L of the exciting circuit. The value of p for the case considered is easily calculated from the ordinary formulæ, and it is found to lie between one-ninth and one-twelfth of L. This would only give sparks of from $\frac{1}{2}$ to $\frac{2}{3}$ millimetre in length, so that according to theory visible sparks ought in any case to be obtained; but, on the other hand, sparks several millimetres in length, as were obtained in the experiments previously described, can only be explained on the assumption that the successive inductive actions produce an accumulative effect; so that theory indicates the necessity of the existence of the resonant effects actually observed.

Dr. Hertz was at first inclined to suppose that as the micrometer circuit was only broken by the extremely short air space limited by the maximum sparking distance under the conditions of the experiment, it might therefore be treated as a closed circuit, and only the total induction considered. The ordinary methods of electro-dynamics give the means of completely determining the total inductive effect of a current element on a closed circuit, and would, therefore, in this case have sufficed for the investigation of the phenomena observed. He found, however, that the treatment of the micrometer circuit as a closed circuit led to incorrect results, so that it, as well as the primary, had to be treated as an open circuit, and therefore a knowledge of the total induction was insufficient, and it became necessary to consider the value both of the E.M.F. of induction and of the electrostatic E.M.F. due to the charged extremities of the exciting circuit at each point of the micrometer circuit.

The investigations to which these considerations led are described by Dr. Hertz in a Paper, "On the Action of a Rectilinear Electrical Oscillation upon a Circuit in its Vicinity," published in Wiedemann's *Annalen*, Vol. XXXIV., p. 155, 1888.

In what follows, the exciting circuit will be spoken of as the primary and the micrometer circuit as the secondary. Hertz points out that the reason that the electrostatic effect cannot be neglected is to be found in the extreme rapidity with which the electrostatic forces change their sign. If the electrostatic alternations in the primary were comparatively slow they might attain a very high intensity without giving rise to a spark in the secondary, since the electrostatic distribution on the secondary would vary so as to remain in equilibrium with the external E.M.F. This, however, is impossible, because the variations in direction follow each other too rapidly for the distribution to follow them.

In the present investigations the primary circuit consisted of a straight copper wire 5 millimetres in diameter, carrying at its extremities hollow zinc spheres 30 centimetres in diameter. The centres of the spheres were one metre apart, and at the middle of the wire was an air space $\frac{3}{4}$ centimetre in length. The wire was placed in a horizontal position, and the observations were all made at points near to the horizontal plane through it, which, however, did not of course affect their generality, as the same effects would necessarily be produced in any plane through the horizontal wire. The secondary circuit consisted of a circle of 35 centimetres radius, of copper wire 2 millimetres in diameter, the circle being broken by an air space capable of variation by means of a micrometer screw.*

The circular form was selected for the secondary circuit because the former investigations had shown that the sparking distance was not the same at all points of the secondary, even when the conductor as a whole remained unchanged in position, and with a circular circuit it was easier to bring the air space to any part than if any other form had been used. To attain this object the circle was made movable about an axis passing through its centre perpendicular to its plane.

The circuits of the dimensions stated were very nearly in unison, and they were further adjusted by means of little strips of metal soldered to the extremities, and varied in length until the maximum sparking distance was obtained.

We shall follow Dr. Hertz in first considering the subject theoretically, and then examining how far the experimental results are in accordance with the theoretical conclusions. It will be assumed that the E.M.F. at every point is a simple harmonic function of the time, but that it does not undergo reversal in direction, and it will further be assumed that the oscillations are at any given moment everywhere in the same phase. This will certainly be the case in the immediate neighbourhood of the primary, and for the present we shall confine our attention to such points. Let s be the distance of a point measured along the circuit from the air space of the secondary, and F the component E.M.F. at that point along the circular arc $d\,s$. Then F is a function of s, which assumes its original

* This small circular detector circuit may be called *an electro-magnetic eye*, because it enables us to see the electro-magnetic disturbance and to localise it.

value after passing once round the circle of circumference S. It may, therefore, be expanded in the form

$$F = A + B\cos\frac{2\pi s}{S} + \ldots\ldots + B'\sin\frac{2\pi s}{S} + \ldots\ldots$$

The higher terms of the series may be neglected, as the only result of so doing will be that the approximate theory will give an absolute disappearance of sparks where really the disappearance is not quite complete, and indeed the experiments are not delicate enough to enable us to compare their results with theory beyond a first approximation.

The force A acts in the same direction, and is of constant amount at all points of the circle, and therefore it must be independent of the electrostatic E.M.F., as the integral of the latter round the circle is zero. A, then, represents the total E.M.F. of induction, which is measured by the rate of variation of the number of magnetic lines of force which pass through the circle. If the electro-magnetic field containing the circle is assumed to be uniform, A will therefore be proportional to the component of the magnetic induction perpendicular to the plane of the secondary. It will therefore vanish when the direction of the magnetic induction lies in the plane of the secondary. A will consist of an oscillation, the intensity of which is independent of the position of the air space in the circle, and the corresponding sparking distance will be called a.

The term $B'\sin\frac{2\pi s}{S}$ can have no effect in exciting the fundamental vibration of the secondary, since it is symmetrical on opposite sides of the air space.

The term $B\cos\frac{2\pi s}{S}$ will give a force acting in the same direction in the two quadrants opposed to the air space, and will excite the fundamental vibration. In the two quadrants adjacent to the air space it will give a force in the opposite direction, but its effect will be less than that of the former one; for the current is zero at the extremities of the circuit, and therefore the electricity cannot move so freely as near the centre. This corresponds to the fact that if a string fastened at each end has its central portion and ends acted on respectively by oppositely directed forces, its motion will be that due to the force at the central portion, which will excite the fundamental vibration if its oscillations are in unison with the latter. The intensity of the vibration will be proportional to B. Let E be the total E.M.F. in the uniform field of the secondary, ϕ the angle between its direction and the plane of the latter, and θ the angle which its projection on this plane makes with the radius drawn to the air space. Then we shall have, approximately,

$$F = E \cos \phi \sin \left(\frac{2 \pi s}{S} - \theta \right),$$

and, therefore, $B = -E \cos \phi \sin \theta$.

B, therefore, is a function simply of the total E.M.F. due both to the electrostatic and electro-dynamic actions. It will vanish when $\phi = 90°$—that is to say, when the total E.M.F. is perpendicular to the plane of the circle, whatever be the position of the air space on the circle. B will also vanish when $\theta = 0$—that is to say, when the projection of the E.M.F. on the plane of the circle coincides with the radius through the air space. If the position of the air space on the circle is varied, the angle θ will vary, and, therefore, also the intensity of the vibration and the sparking distance. The sparking distance corresponding to the second term of the expansion for F can therefore be represented approximately by a formula of the form $\beta \sin \theta$.

Now, the oscillations giving rise to sparks of lengths α and $\beta \sin \theta$ respectively are in the same phase. The resulting oscillations will therefore be in the same phase, and their amplitudes must be added together. The sparking distance being approximately proportional to the maximum total amplitude may therefore also be obtained by adding the sparking distances due to the two oscillations respectively. The sparking distance will therefore be given as a function of the position of the air space on the secondary circuit by the expression $\alpha + \beta \sin \theta$. Since the direction of the oscillation in the air space does not come into consideration, we are concerned only with the absolute value of this expression and not with its sign. The determination of the absolute values of the quantities α and β would involve elaborate theoretical investigations, and is, moreover, unnecessary for the explanation of the experimental results.

Experiments with the Secondary Circuit in a Vertical Plane.—When the circle forming the secondary circuit was placed with its plane vertical, anywhere in the neighbourhood of the primary, the following results were obtained :—

The sparks disappeared for two positions of the air space, separated by 180 deg., namely, those in which it lay in the horizontal plane through the primary ; but in every other position sparks of greater or less length were observed.

From this it followed that the value of α must have been constantly zero, and that θ was zero when the air space was in the horizontal plane through the primary.

The electro-magnetic lines of force must therefore have been perpendicular to this horizontal plane, and therefore consisted of circles with their centres on the primary ; while the electrostatic lines of force must have been entirely in the horizontal plane, and therefore this system of lines of force consisted of curves lying in planes passing through the primary. Both of these results are in agreement with theory.

When the air space was at its greatest distance from the plane the sparking distance attained a maximum value of from 2 to 3 millimetres. The sparks were shown to be due to the fundamental vibration, by slightly varying the secondary so as to throw it out of unison with the primary, when the sparking distance was diminished, which would not have been the case if the sparks had been due to overtones. Moreover, the sparks disappeared when the secondary was cut at its points of intersection with the horizontal plane through the primary, though these would be nodal points for the first overtone.

When the air space was kept at its greatest possible distance from the horizontal plane through the primary, and turned about a vertical axis, the sparking distance attained two maxima at the points for which $\phi = 0$, and almost disappeared at the points for which $\phi = 90°$.

The lower half of Fig. 31 shows the different positions of minimum sparking. A A′ is the primary conductor, and the lines $m\ n$ represent the projections of the secondary circuit on the horizontal plane. The arrows perpendicular to these give the direction of the resultant lines of force. As this did not anywhere vanish in passing from the sphere A to the sphere A′, it could not change its sign.

The diagram brings out the two following points :—

1. The distribution of the resultant E.M.F. in the vicinity of the rectilinear vibration is very similar to that of the electrostatic E.M.F. due to the action of its two extremities. It should be specially noted that near the centre of the primary the direction is that of the electrostatic E.M.F., showing that it is more powerful than the electro-dynamic, as required by theory.

2. The lines of force deviate more rapidly from the line A A′ than the electrostatic lines, though this is not so evident on the reduced scale of the diagram as in the author's original drawings on a much larger scale.

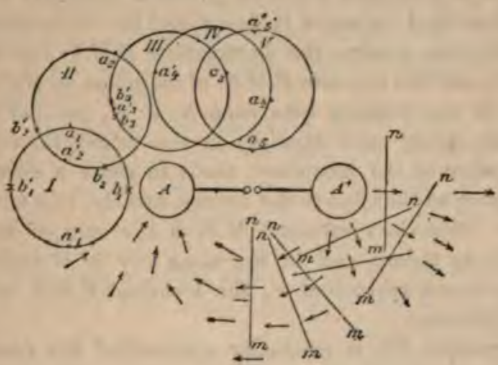

Fig. 31.

It is due to the components of the electrostatic E.M.F. parallel to A A′ being weakened by the E.M.F. of induction, while the perpendicular components remain unaffected.

Experiments with the Secondary Circuit in a Horizontal Plane. —The results obtained when the plane of the secondary was horizontal can best be explained by reference to the upper half of the diagram in Fig. 31.

In the position I., with the centre of the circle in the line A A′ produced, the sparks disappeared when the air space occupied either of the positions b_1 or b'_1, while two equal maxima of the sparking distance were obtained at a_1 and a'_1, the length of the spark in these positions being 2·5 millimetres. Both these results are in accordance with theory.

In the position II. the circle is cut by the electro-magnetic lines of force, and therefore α does not vanish. It will, however, be small, and we should expect that the expression $\alpha + \beta \sin \theta$ would have two unequal maxima, $\beta + \alpha$ and $\beta - \alpha$, both for $\theta = 90^{\circ}$, and having the line joining them perpendicular to the resultant E.M.F., and between these two maxima we should expect two points of no sparking near to the smaller maximum. This was confirmed by the observations.

The maximum sparking distances were 3·5 millimetres at a_2 and 2 millimetres at a'_2. Now, with the air space at a_2, the sphere A being positive, the resultant E.M.F. in the opposite portion of the circle will repel positive electricity from A, and therefore tend to make it flow round the circle clockwise. Between the two spheres the electrostatic E.M.F. acts from A towards A′, and the opposite E.M.F. of induction in the neighbourhood of the primary acts from A′ to A, parallel to the former, and, acting more strongly on the nearer than on the further portion of the secondary, tends to cause a current in same direction as that due to the former, namely, in a clockwise direction. Thus the resultant E.M.F. is the sum of the two as required by theory, and in the same way it is easily seen that when the air space is at a'_2 the resultant E.M.F. is equal to their difference.

As the position III. is gradually approached the maximum disappears, and the single maximum sparking distance a_3 was found to be four millimetres in length, having opposite to it a point of disappearance a'_3. In this case clearly $\alpha = \beta$, and the sparking distance is given by the expression $\alpha\,(1 + \sin \theta)$. The line $a_3\, a'_3$ is again perpendicular to the resultant E.M.F.

As the circle approaches further towards the centre of A A′, α will become greater than β, and the expression $\alpha + \beta \sin \theta$ will not vanish for any value of θ, but will have a maximum $\alpha + \beta$ and a minimum $\alpha - \beta$; and in the experiments it was found that the sparks never entirely disappeared, but varied between a maximum and a minimum, as indicated by theory.

In the position IV. a maximum sparking distance of 5·5 millimetres was observed at a_4, and a minimum of 1·5 millimetre at a'_4.

In the position V. there was a maximum sparking distance of 6 millimetres at a_5, and a minimum of 2·5 millimetres at a'_5. In these experiments the air space should be screened off from the primary in the latter positions as well as in the earlier ones, in which it is unavoidable, as otherwise the results would not be comparable.

In passing from the position III. to the position V. the line $a\ a'$ rapidly turned from its position of parallelism to the primary circuit into a position perpendicular to it. In the latter positions the sparking was essentially due to the inductive action, and therefore the author was justified, in the experiments described in my previous Papers, in assuming the effect in these positions to be due to induction.

Even in these positions, however, the sparking is not totally independent of electrostatic action, except when the air space is half-way between the maximum and minimum positions, and, therefore $\beta \sin \theta = 0$.

Other Positions of the Secondary Circuit.—Dr. Hertz made numerous observations with the secondary circuit in other positions, but in no case were any phenomena observed which were not completely in accordance with theory. As an example of these consider the following experiment :—

The secondary was first placed in the horizontal plane in the position V. (Fig. 31), and the air space was in the position a_5 relatively to the primary. The circle was then turned about a horizontal axis through its centre and parallel to the primary, so as to raise the air space above the horizontal plane. During this rotation θ remained equal to 90deg., and the value of β remained nearly constant, but α varied approximately in the same ratio as $\cos \Psi$, Ψ, being the angle between the plane of the circle and the horizontal, for α is proportional to the number of magnetic lines of force passing through the circle. Let α_0 be the value of α in the initial position, then in the other positions its value would be $\alpha_0 \cos \Psi$, and therefore the sparking distance should be given by the expression $\alpha_0 \cos \Psi + \beta$, in which α_0 was known to be greater than β. This was confirmed by observation, for it was found that as the air space increased its height above the horizontal plane the sparking distance diminished from 6 millimetres down to 2 millimetres, its value when the air space was at its greatest distance above the horizontal plane. During the rotation through the next quadrant the sparking distance diminished almost to zero, and then increased to the smaller maximum of 2·5 millimetres, which it attained when the circle had turned through 180deg., and was therefore again horizontal. Similar results were obtained in the opposite order as the circle was rotated from 180deg. to 360deg. When the circle was kept with the air space at its maximum height above the horizontal plane, and then raised or lowered bodily without rotation, the sparking distance was found to diminish in the former case and to increase in the latter—results completely in accordance with theory.

Forces at Greater Distances.—Experiments with the secondary at greater distances from the primary are of great importance, as the distribution of E.M.F. in the field of an open circuit is very different according to different theories of electro-dynamic action, and the results may, therefore, serve to eliminate some of them as untenable. In making these experiments, however, an unexpected difficulty was encountered, as it was found that at distances of from 1 to 1·5 metre from the primary, the maximum and minimum, except in certain positions, became indistinctly defined; but when the distance was increased to upwards of two metres, though the sparks were then very small, the maximum and minimum were found to be very sharply marked when the sparks were observed in the dark. The positions of maximum and minimum were found to occur with the circle in planes at right angles to each other. At considerable distances the sparking diminished very slowly as the distance was increased. Dr. Hertz was not able to determine an upper limit to the distance at which sensible effects took place, but, in a room 14 metres by 12, sparks were distinctly observed when the primary was placed in one corner of the room wherever the secondary was placed. When, however, the primary was slightly displaced no effects could be observed, even when the secondary was brought considerably nearer. The interposition of solid screens between the two circuits greatly diminished the effect.

Dr. Hertz mapped out the distribution of force throughout the room by means of chalk lines on the floor, putting stars at the points where the direction of the E.M.F. became indeterminate. A portion of the diagram obtained in this manner is shown on a reduced scale in Fig. 32, with respect to which the following points are noteworthy :—

1. At distances beyond three metres the E.M.F. is everywhere parallel to the primary oscillation. Within this region, therefore, the electrostatic E.M.F. is negligible in comparison with the E.M.F. of induction. Now all the theories of the mutual action of current elements agree in giving an E.M.F. of induction inversely proportional to the distance; while the electrostatic E.M.F., being due to the differential action of the two extremities of the primary, is approximately inversely proportional to the cube of the distance. Some of these theories, however, are not in accordance with the experimental result that the effect diminishes much more rapidly

FIG. 32.

in the direction of the primary oscillation than in a direction at right angles to it, induced sparks being observed at a distance exceeding 12 metres in the latter direction, while they disappeared at a distance of about four metres in the former direction.

2. That, as already proved, for distances less than one metre, the distribution of E.M.F. is practically that of the electrostatic E.M.F.

3. There are two straight lines at all points of which the direction of the E.M.F. is determinate, namely, the line in which the primary oscillation takes place, and the perpendicular to the primary through its middle point. Along the latter the E.M.F. does not vanish at any point; the sparking diminishes gradually as the distance is increased. This, again, is incon-

sistent with some of the theories of mutual action of current elements, according to which it should vanish at a certain definite distance. A very important result of the investigation is the demonstration of the existence of regions within which the direction of the E.M.F. becomes indeterminate. These regions form two rings encircling the primary circuit. Since the E.M.F. within them acts very nearly equally in every direction, it must assume different directions in succession, for, of course, it cannot act in different directions simultaneously.

The observations, therefore, lead to the conclusion that within these regions the magnitude of the E.M.F. remains very nearly constant, while its direction varies through all the points of the compass at each oscillation. Dr. Hertz states that he has been unable to explain this result, as also the existence of overtones, by means of the simplified theory in which the higher terms of the expansion of F are neglected, and he considers that no theory of simple action at a distance is capable of explaining it. If, however, the electrostatic E.M.F. and the E.M.F. of induction are propagated through space with unequal velocities, it admits of very simple explanation. For within these annular regions the two E.M.F.s are at right angles and of the same order of magnitude; they will, therefore, in consequence of the distance traversed, differ in phase, and the direction of the resultant will turn through all the points of the compass at each oscillation.

This phenomenon appears to him to be the first indication which has been observed of a finite rate of propagation through space of electrical actions, for, if there is a difference in the rate of propagation of the electrostatic and electro-dynamic E.M.F., one at least of them must be finite.

At the end of the Paper in which the preceding experiments are described, Dr. Hertz describes some observations which he has made on the conditions at the primary sparking point which affect the production of sparks in the secondary circuit. He finds that illuminating the primary spark diminishes its power of exciting rapid oscillations, the sparks in the secondary being observed to cease when a piece of magnesium wire was burnt or an arc lamp lighted near the primary point. The observed effect on the primary sparks is that they are no longer accompanied by a sharp crackling sound as before. The effect of a second discharge is especially noteworthy, and it was found that the secondary sparks could be made to disappear by bringing an insulated conductor close to the opposed surfaces of the spheres forming the terminals at the primary air space, even when no visible sparking took place between the latter and the insulated conductor. The secondary sparking could also be stopped by placing a fine point close to the primary air space, or by touching one of the opposed surfaces of the terminals with a piece of sealing-wax, glass, or mica. Dr. Hertz states that further experiments have led him to conclude that, even in these cases, the effect is due to light too feeble to be perceived by the eye, arising from a side discharge. He points out that these effects afford another example of the effects of light on electric discharges, which have been observed by E. Wiedemann, H. Hebert, and W. Hallwachs.

Dr. Hertz's next Paper in order of publication in Wiedemann's *Annalen* is "On Some Induction Phenomena Arising from Electrical Actions in Dielectrics" (Vol. XXXIV., p. 273), and contains an account of some researches which were undertaken with a view of obtaining direct experimental confirmation of the assumption involved in the most suggestive theory of electrical actions, viz., that of Faraday and Maxwell, that the well-known electrostatic phenomena observed in dielectrics are accompanied by corresponding electro-dynamic actions. The method of observation consisted in placing a secondary conductor adjusted to unison, as regards electrical oscillations, with the primary, as near as possible to the former, and in such a relative position that the sparks in the primary produced no sparking in the secondary. As the equilibrium could be disturbed and sparking induced in the secondary by the approach of conductors, it formed a kind of induction balance; but the point of special interest in connection with it was that a similar effect was produced when the conductors were replaced by insulators, provided the latter were of comparatively large size. The observed rapidity of the oscillations induced in the dielectrics showed that the quantities of electricity in motion under the influence of dielectric polarisation were of the same order of magnitude as in the case of metallic conductors.

The apparatus employed is shown diagrammatically in Fig. 33, and was supported on a light wooden framework, not shown in the illustration. The primary conductor consisted of two brass plates, A A′, with sides 40 centimetres in length, joined by a copper wire 70 centimetres long and half a centimetre in diameter, containing an air space of three-quarters of a centimetre, with terminals formed of polished brass spheres. When placed in connection with a powerful induction coil, oscillations are set up, the period of which, determined by the dimensions

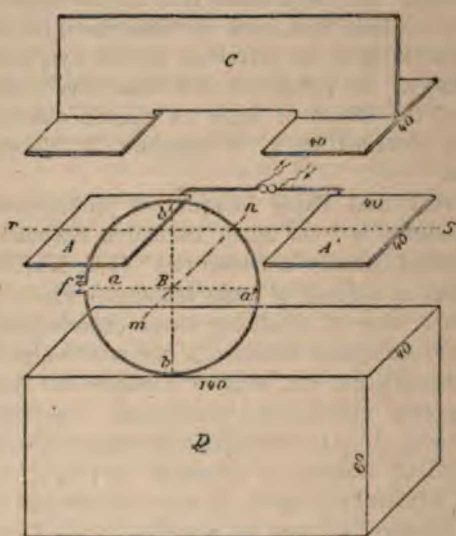

Fig. 33.

of the primary, can be determined to within a hundred-millionth of a second. The secondary conductor consisted of a circle, 35 centimetres in radius, of copper wire two millimetres in diameter, containing an air space, the length of which could be varied by means of a screw from a few hundredths of a millimetre up to several millimetres. The dimensions stated were such as to bring the two conductors into unison, and secondary sparks up to six or seven millimetres in length could be obtained.

The circle was movable about an axis through its centre perpendicular to its plane, to enable the position of the air space to be varied. The axis was fixed in the position *m n* in the plane of A and A′, and half-way between them. The centre of the circle was at a distance of 12 centimetres from the nearest points of A and A′.

When *f* was in either of the positions *a* or *a′* lying in the plane of A A′ no sparking occurred in the secondary, while maximum sparking took place at *b* and *b′* 90deg. from the former positions. The E.M.F. giving rise to the secondary sparks is, as in previous experiments, partly electrostatic and partly electro-magnetic, and the former being the greater will determine the sign of the resultant E.M.F. The oscillations must, for the reason previously explained, be considered as produced in the part of the secondary most remote from the air space. Assuming the E.M.F. and the amplitude of the resulting oscillation to be positive when *f* is in the position *b′*, they will both be negative when *f* is at *b*.

When the circle was slightly lowered in its own plane the sparking distance was increased at *b′* and diminished at *b*, and the null points lay at a certain distance below *a* and *a′*. The electrostatic E.M.F. is scarcely affected by such a displacement, but the integral of the E.M.F. of induction taken round the circle is no longer zero, and therefore gives rise to an oscillation which will be of positive sign whatever be the position of *f*, for the direction of the resultant E.M.F. of induction is opposite to that of the electrostatic E.M.F. in the upper half of the circle, and coincides with it in the lower half, where the electrostatic E.M.F. has been assumed to be positive. Since the new oscillation so produced is in the same phase as the previously existing one, their amplitudes must be added to give the resultant amplitude, which explains the phenomena.

Effects of the Approach of Conductors.—In making these observations it was found necessary to remove all conductors to a considerable distance from the apparatus, in order to obtain a complete disappearance of sparking at the points *a* and *a′*. Even the neighbourhood of the observer was sufficient to set up sparking when the air space *f* was in either of these positions, and the sparks had therefore to be observed from a distance. The conductors used for the experiments were of the form shown at C (Fig. 10), and consisted of thin metal foil. The objects kept in view in selecting the material and dimensions were to obtain a conductor which would give a moderately large effect and having an oscillation period less than that of the primary.

When the conductor C was brought near to A A′, it was found that the sparking distance decreased at b and increased at b', and the null points were displaced upwards—that is, in the direction of C.

From the results of experiments already described it is evident that the effect of displacing A A′ upwards would be the same, qualitatively, as that of a current in the same direction as that in A A′ directly above it. The effect produced by the approach of C was the reverse of this, and could be explained by an inductive action, supposing there were a current in C in the opposite direction to that in A A′, which is exactly what must occur; for the electrostatic E.M.F. would give rise to such a current, and since the oscillations in C are more rapid than those of the E.M.F., the current must be in the same phase as the inducing E M.F. The truth of this explanation was confirmed by the following experiments. The horizontal plates of the conductor C being left in the same position as before, the vertical plate was removed, and successively replaced by wires of increasing length and fineness, in order to lengthen the oscillation period of C. The effect of this was to displace the null points more and more in an upward direction, while at the same time they became less sharply defined, a minimum sparking taking the place of the previous absolute disappearance. The sparking distance at the highest point had previously been much less than at the lowest point, but after the disappearance of the null points it began to increase. At a certain stage the sparking distance at the two positions became equal, and then no definite minimum points could be found, but sparking took place freely at all positions of f. Beyond this stage the sparking distance at the lowest point diminished, and very soon two minimum points made their appearance close to it, not clearly defined at first, but gradually becoming more distinct, and at the same time approaching the points a a', with which they ultimately coincided, when the minimum points again became absolute null points. These results are in agreement with the conclusion drawn from the former observations, for as the oscillation period of C approaches that of A A′ the intensity of the current in the former increases, but a difference of phase arises between it and the existing E.M.F. When the two are in unison the current in C attains its maximum, and, as in other cases of resonance, the difference of phase gives rise to a slightly damped oscillation, having a period of about a quarter that of the original one, which makes any interference between the oscillations excited in the circle B by A A′ and C respectively impossible. These conditions clearly correspond to the stage at which the sparking distances at *b* and *b*′ were equal. When the oscillation period of C becomes decidedly greater than that of A A′, the amplitude of the oscillation in the former will again diminish, so that the difference in phase between it and the exciting E.M.F. will approach half of the original period. The current in C will therefore always be in the same direction as that in A A′, so that interference between the two oscillations excited in B will again become possible, and the effect of C will then be opposite to its original effect. When the conductor C was made to approach A A′ the sparks in B became much smaller, which is explained by the fact that its effect will be to increase the oscillation period of A A′, and therefore to throw it out of unison with B.

Effects of the Approach of Dielectrics.—A very rough estimate shows that when a dielectric of large mass is brought near to the apparatus the quantities of electricity set in motion by dielectric polarisation are at least as large as in metallic wires or thin rods. If, therefore, the action of the apparatus were unaffected by the approach of such masses, it would show that, in contradiction to the theories of Faraday and Maxwell, no electro-dynamic actions are called into play by means of dielectric polarisation, or, as Maxwell calls it, electric displacement. The experiments, however, showed an effect similar to that which would be produced if the dielectric were replaced by a conductor with a very small oscillation period. In the first experiment made, the mass of dielectric consisted of a pile of books, 1·5 metre long, 0·5 metre broad, and 1 metre high, placed under the plates A A′. Its effect was to displace the null points through about 10deg. towards the pile. A block of asphalte (D, in Fig. 33), weighing 800 kilogrammes, and measuring 1·4 metre in length, 0·4 metre in breadth, and

0·6 metre in height, was then used in place of the books, the plates being allowed to rest upon it.

The following results were then obtained :—

1. The spark at the highest point of the circle was now decidedly stronger than that at the lowest point, which was nearer to the asphalte.

2. The null points were displaced through about 23deg. downwards—that is, in the direction of the block—and at the same time were transformed into mere points of minimum sparking, a complete disappearance being no longer obtainable.

3. When the plates A A′ rested on the asphalte block the oscillation period of the primary was increased, as shown by the fact that the period of B had to be slightly increased in order to obtain the maximum sparking distance

4. When the apparatus was moved gradually away from the block its action steadily diminished without changing its character.

5. The action of the block could be compensated by bringing the conductor C over the plates A A′ while they rested on the block, the null points being brought back to a and a' when C was at a height of 11 centimetres above the plates. When the upper surface of the asphalte was 5 centimetres below the plates, compensation was obtained when C was placed at a height of 17 centimetres above them, showing that the action of the dielectric was of the order of magnitude which had been anticipated.

The asphalte contained about 5 per cent. of aluminium and iron compounds, 40 per cent. of calcium compounds, and 17 per cent. of quartz sand. In order to make sure that the observed effects were not due to the conductivity of some of these substances a number of further experiments were made.

In the first place, the asphalte was replaced by a mass of the same dimensions of the so-called artificial pitch prepared from coal, and effects of a similar kind were observed, but slightly weaker, the greatest displacement of the null points amounting to 19deg. Unfortunately this pitch contains free carbon, the amount of which it is difficult to determine, and this would have some conductivity.

The experiments were then repeated with a conductor, C, of half the linear dimensions of the former one, and smaller blocks of various substances, on account of the great cost of obtaining large blocks of pure materials. The substances used were asphalte, coal-pitch, paper, wood, sandstone, sulphur, paraffin, and also a fluid dielectric, namely, petroleum. With the smaller apparatus it was not possible to obtain quantitative results of the same accuracy as before, but the effects were of an exactly similar character, and left little room for doubt of the reality of the action of the dielectric.

The results might possibly be supposed to be due to a change in the distribution of the electrostatic E.M.F. in the neighbourhood of the dielectric, but, in the first place, Dr. Hertz states that he has been unable to explain the details of the observations on this hypothesis, and in the second place it is disproved by the following experiment:—

The smaller apparatus was placed with the line *r s* on the upper near corner of one of the large blocks, in which position the dielectric was bounded by the plane of the plates A A′ and the perpendicular plane through *r s*, both of which are equipotential surfaces, so that if the action were electrostatic no effect should be produced by the dielectric. It was found, however, to produce the same effect as in other positions. It might also be supposed that the effects were due to a slight conductivity, but this could hardly be the case with such good insulators as sulphur and paraffin. Suppose, moreover, that the conductivity of the dielectric is sufficient to discharge the plate A in the ten-thousanth of a second, but not much more rapidly; then, during one oscillation, the plates would lose only the ten-thousandth part of their charge, and the conduction current in the substance experimented on would not exceed the ten-thousandth part of the primary current in A A′, so that the effect would be quite insensible.

It is thus shown in the experiments described above that when variable electrical forces act in the interior of dielectrics of specific inductive capacity not equal to unity the corresponding electric displacements produce electro-dynamic effects. In a Paper, "On the Velocity of Propagation of Electro-Dynamic Actions," in Wiedemann's *Annalen*, Vol. XXXIV., p. 551, Dr. Hertz shows that similar actions take place in the air, which proves, as was previously pointed out, that electro-dynamic action must be propagated with a finite velocity.

The method of investigation was to excite electrical oscillations in a rectilinear conductor in the same manner as in former experiments, and then to produce effects in a secondary conductor by exciting electrical oscillations in it by means of those in the rectilinear conductor, and at the same time by the primary conductor acting through the intervening space. This distance was gradually increased, when it was found that the phase of the vibrations at a distance from the primary lagged behind those in its immediate neighbourhood, showing that the action is propagated with a finite velocity which was found to be greater than the velocity of propagation of electrical waves in wires in the ratio of about 45 to 28, so that the former

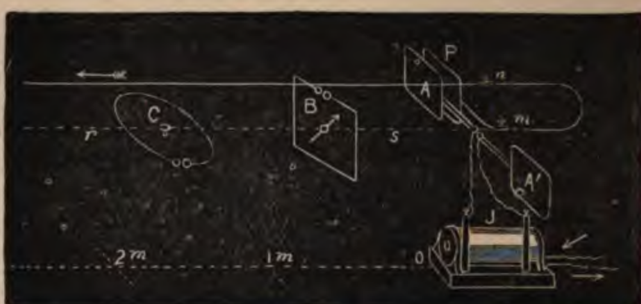

FIG. 34.

is of the same order as the velocity of light. Dr. Hertz was unable to obtain any evidence with respect to the velocity of propagation of electrostatic actions.

The primary conductor A A′ (Fig. 34) consisted of a pair of square brass plates with sides 40 centimetres in length, connected by a copper wire 60 centimetres in length, at the middle point of which was an air space, across which sparks were made to pass by means of powerful discharges from the induction coil J. The conductor was fixed at a height of 1·5 metre above the base-plate of the coil, with its plates vertical, and the connecting wire horizontal. A straight line, *r s*, drawn horizontally through the air space of the primary, and perpendicular to the direction of the primary oscillation, will be called "the base-line;" and a point in this, situated at a distance of 45 centimetres from the air space, will be referred to as "the null point."

The experiments were made in a large lecture-room, with nothing near the base-line for a distance of 12 metres from the primary conductor. The room was darkened during the experiments.

The secondary conductor consisted either of a circular wire, C, of 35 centimetres radius, or of a square of wire, B, with sides 60 centimetres long. The primary and secondary air spaces were both capable of adjustment by means of micrometer screws. Both the secondary conductors were in unison with the primary, the (half) vibration period of each being 1·4 hundred-millionth of a second, as calculated from the capacity and coefficient of self-induction. It is doubtful whether the ordinary theory of electrical oscillations would lead to accurate results under the conditions of these experiments, but as it gives correct numerical results in the case of Leyden jar discharges, it may be expected to be correct as far as the order of the results is concerned. When the centre of the secondary lies in the base-line, and its plane coincides with the vertical plane through the base-line, no sparks are observed in the secondary, the E.M.F. being everywhere perpendicular to the direction of the secondary. This will be referred to as "the first principal position" of the secondary. When the plane of the secondary is vertical and perpendicular to the base-line, the centre still lying in the base-line, the secondary will be said to be in its "second principal position." Sparking then occurs in the secondary when its air space is either above or below the horizontal plane through the base-line, but not when it is in this plane. As the distance from the primary was increased, the sparking distance was observed to decrease, rapidly at first, but ultimately very slowly. Sparks were observed throughout the whole distance of 12 metres available for the experiments. The sparking in this position is due essentially to the E.M.F. produced in the portion of the secondary remote from the air space. The total E.M.F. is partly electrostatic and partly electro-dynamic, and the experiments show beyond the possibility of doubt that the former is greater, and therefore determines the direction of the total E.M.F. close to the primary, while at greater distances it is the electro-dynamic E.M.F. which is the greater.

The plane of the secondary was then turned into the horizontal, its centre still lying in the base-line. This may be called "the third principal position." When the centre of the circular secondary conductor was kept fixed at the null point, and the air space was made to travel round the circle, vigorous sparking was observed in all positions. The sparking distance attained its maximum length of about six millimetres when its air space was nearest to that of the primary, and its minimum length of about three millimetres when the distance between the two air spaces was greatest. If the secondary had been influenced by the electrostatic force, sparking would only be expected when the air space was close to the base-line, and a cessation of sparks in the intermediate positions. The direction of the oscillation would, moreover, be determined by the direction of the E.M.F. in the portion of the secondary furthest from the air space. There is, however, superposed upon the electrostatically excited oscillation a second oscillation, due to the E.M.F. of induction, which produces a considerable effect, since its integral round the circle (considered as a closed circuit) does not vanish; and the direction of this integral E.M.F. is independent of the position of the air space, opposing the electrostatic E.M.F. in the portion of the secondary next to A A′, and assisting it in the portion furthest from A A′, as explained in a previous Paper.

The electrostatic and electro-dynamic E.M.F.s, therefore, act in the same direction when the air space is turned towards the primary conductors, and in opposite directions when the air space is turned away from the primary. In the latter position it is the E.M.F. of induction which is the more powerful, as is shown by the fact that there is no disappearance of sparking in any position of the air space, for when this is 90deg. to the right or left of the base-line it coincides with a node with respect to the electrostatic E.M.F. In these positions the inductive action in the neighbourhood of the primary can be observed independently of the electrostatic action.

Waves in Rectilinear Wires.—In order to produce in a wire by means of the primary oscillations a series of advancing waves of the character required for these experiments, the following arrangements were made:—Behind the plate A was placed a plate, P, of equal size. A copper wire one millimetre in diameter connected P to the point M of the base-line. From M the wire was continued in a curve about a metre in length to the point N, situated about 30 centimetres above the air space, and was then further continued in a straight line parallel to the base-line for such a distance as to obviate all danger of disturbance from reflected waves. In the present series of experiments the wire passed through a window, and after being carried to a distance of about 60 metres, was put to earth, and a special series of experiments showed that this length was sufficient. When a wire, bent so as to form a nearly closed circuit with a small air space, was brought near to this straight wire, a series of fine sparks was seen to accompany the discharges of the induction coil. Their intensity could be varied by varying the distance between the plates P and A. The waves in the rectilinear wire were of the same period as that of the primary oscillations, as was proved by their being shown to be in unison with each of the two secondary conductors previously described. The existence of stationary waves showed that the waves in the rectilinear wire were of a steady character in space as well as in time. The nodal points were determined in the following manner:—The further end of the wire was left free, and the secondary conductor was brought near to it in such a position that the wire lay in its plane, and had the air space turned towards it. As the secondary was moved along the wire, points of no sparking were observed to recur periodically. The distance from the point n to the first of these was measured, and the length of the wire made equal to a multiple of this distance. The experiments were then repeated, and it was found that the nodal points occurred at approximately equal intervals along the wire.

The nodes could also be distinguished from the loops in other ways. The secondary conductor was brought near to the wire, with its plane perpendicular to it, and with its air space neither directed completely towards the wire nor completely away from it, but in an intermediate position, so as to produce E.M.F.s perpendicular to the wire. Sparks were then observed at the nodes, while they disappeared at the loops. When sparks were taken from the rectilinear wire by means of an insulated conductor, they were found to be stronger at the nodes than at the loops; the difference, however, was small, and was, indeed, scarcely distinguishable unless the position of the nodes and loops was previously known. The reason that this and other similar methods do not give a well-defined result lies in the fact that irregular oscillations are superposed upon the waves considered; the regular waves, however, can be picked out by means of the secondary, just as definite notes are picked out by means of a Helmholtz resonator. If the wire is severed at a node, no effect is produced upon the waves in the portion of wire next to the origin; but if the severed portion of wire is left in its place the waves continue to be propagated through it, though with somewhat diminished strength.

The possibility of measuring the wave-lengths leads to various applications. If the copper wire hitherto used is replaced by one of different diameter, or by a wire of some other metal, the nodal points retain their position unchanged. It follows from this that the velocity of propagation in a wire has a definite value independent of its dimensions and material. Even iron wires offer no exception to this, showing that the magnetic susceptibility of iron does not play any part in the case of such rapid motions. It would be interesting to investigate the behaviour of electrolytes in this respect. In their case we should expect a smaller velocity of propagation, because the electrical motions are accompanied by motions of the molecules carrying the electric charges. It was found that no propagation of the waves took place through a tube 10 millimetres in diameter, filled with a solution of sulphate of copper; but this may have been due to the resistance being too high. By the measurement of wave-lengths the relative vibration periods of different primary conductors can be determined, and it therefore becomes possible to compare in this manner the vibration periods of plates, spheres, ellipsoids, &c.

In the experiments made by Dr. Hertz, nodes were very distinctly produced when the wire was severed at a distance of either 8 metres or 5·5 metres from the null point of the base-line. In the first case the nodes occurred at distances from the null point of $-0{\cdot}2$ metre, 2·3 metres, 5·1 metres, and 8 metres, and in the latter case at distances of $-0{\cdot}1$ metre, 2·8 metres, and 5·5 metres. It appears, therefore, that the (half) wave-length in a free wire cannot differ much from 2·8 metres. The fact that the wave-lengths nearest to P were somewhat smaller was to be expected from the influence of the plates and of the curvature of the wire. This wave-length, with a period of 1·4 hundred-millionth of a second, gives 200,000 kilometres per second for the velocity of propagation of electrical waves in wires. Fizeau and Gounelle (Poggendorff's *Annalen*, Vol. LXXX., p. 158, 1850) obtained for the velocity in iron wires 100,000 kilometres per second, and 180,000 in copper wires. W. Siemens (Poggendorff's *Annalen*, Vol. CLVII., p. 309, 1876), by the aid of Leyden jar discharges, obtained a velocity of from 200,000 to 260,000 kilometres per second in iron wires. Dr. Hertz's result is very nearly the mean of these, from which we may conclude that the order, at any rate, of the vibration period as calculated by him is correct. The value obtained cannot be regarded, independently of its agreement with experimental results otherwise obtained, as a fresh determination of the velocity, since it rests upon a theory which is open to doubt.

Interference of the Direct Actions with those Transmitted through the Wire.—If the square circuit B is placed at the null point in the second principal position, with the air space at its highest point, it will be unaffected by the waves in the wire, but the direct action when in this position was found to produce sparks 2 millimetres in length. B was then turned about a vertical axis into the first principal position, in which there would be no direct action of the primary oscillation, but the waves in the wire gave rise to sparks, and by bringing P near enough to A a sparking distance of 2 millimetres could be obtained. In the intermediate positions sparks were produced in both these ways, and it would therefore be possible to get a difference of phase, such that one should either increase or diminish the effect of the other. Phenomena of this nature were, indeed, observed. When the plane of B was in such a position that the normal drawn towards A A′ was directed away from that side of the primary conductor on which P was placed, there was more sparking than even in the principal position; but if the normal were directed towards P the sparks disappeared, and only reappeared when the air space was made smaller. When the air space was at the lowest point of B, the other conditions remaining the same, the sparks disappeared when the normal was turned away from P. Further variations of the experiment gave results in accordance with these.

It is easily seen that these phenomena were exactly what were to be expected. To fix the ideas, suppose the air space to be at the highest point, and the normal directed towards P, as in Fig. 34. Consider what happens at the moment that the plate A has its greatest positive charge. The electrostatic, and therefore the total, E.M.F. is directed from A towards A′. The oscillation to which this gives rise in B is determined by the direction of the E.M.F. in the lower portion of B. Therefore positive electricity will flow towards A′ in the lower portion, and away from A′ in the upper portion.

Consider next the action of the waves. As long as A is positively charged, positive electricity will flow from the plate P. This current is at the moment considered at its maximum value at the middle point of the first half wave-length. A quarter of a wave-length further from the origin—that is to say, in the neighbourhood of the null point—it first changes its direction. The E.M.F. of induction will here, therefore, impel positive electricity towards the origin. A current will therefore flow round B towards A′ in the upper portion and away from A′ in the lower portion. The electrostatic and electrodynamic E.M.F.s are therefore in opposite phases and oppose each other's action. If the secondary circuit is rotated through 90deg., through the first principal position, the direct action changes its sign, but not so the action of the waves, so that they now tend to strengthen each other. The same reasoning holds when the air space is at the lowest point of B.

Greater lengths of wire were then included between *m* and *n*, and it was found that the interference became gradually less marked, until within a length of 2·5 metres it disappeared entirely, the sparks being of equal length whether the normal were directed towards or away from P. When the length of wire between *m* and *n* was further increased, the distinction between the different quadrants reappeared, and with a length of 4 metres the disappearance of the sparks was fairly sharp. The disappearance, however, then took place (with the air space at the highest point) when the normal was directed away from P, the opposite direction to that in which the disappearance previously took place. With a still further increase in the length of the wire the interference reappeared, and returned to its original direction with a length of 6 metres. These phenomena are clearly to be explained by the retardation of the waves in the wire, and show that here again the direction of motion in the advancing waves changes its sign at intervals of about 2·8 metres.

To obtain interference phenomena with the secondary circuit C in the third principal position, the rectilinear wire must be removed from its original position and placed in the horizontal plane through C either on the side of the plate A or of the plate A′. Practically it is sufficient to stretch the wire loosely, and to fix it by means of an insulated clamp on each side of C alternately. It was found that when the wire was on the same side as the plate P the waves in it diminished the previous sparking, and when on the opposite side the sparking was increased, both results being unaffected by the position of the air space in the secondary circuit. Now it has been already pointed out that at the moment when the plate A has its maximum positive charge, and at which, therefore, the primary current begins to flow from A, the current at the first node of the rectilinear wire begins to flow away from the origin. The two currents, therefore, flow round C in the same direction when C lies between the rectilinear wire and A, and in opposite directions when the wire and A are on the same side of C. The fact that the position of the air space is indifferent confirms the conclusion formerly arrived at that the direction of oscillation is that due to the electro-dynamic E.M.F. These interferences are also changed in direction when the wire $m\,n$, 1 metre in length, is replaced by a wire 4 metres in length.

Dr. Hertz also succeeded in obtaining interference phenomena when the centre of the secondary circuit was not in the base-line, but these results were of no special importance, except that they confirmed the previous conclusions.

Interference Phenomena at Various Distances.—Interference may be produced with the secondary at greater distances than that of the null point; but care must then be taken that the action of the waves in the wire is of about the same magnitude as the direct action of the primary circuit through the air. This can be effected by increasing the distance between P and A.

Now, if the velocity of propagation of the electro-dynamic disturbances through the air is infinite, the interference will change its sign at every half-wave length in the wire—that is to say, at intervals of about 2·8 metres. If the velocities of propagation through the air and through the wire are equal, the interference will be in the same direction at all distances. Finally, if the velocity of propagation through the air is finite, but different from the velocity in the wire, the interference will change in sign at intervals greater than 2·8 metres.

The interferences first investigated were those which occurred when the secondary circuit was rotated from the first into the second principal position, the air space being at the highest point. The distance of the secondary from the null point was increased by half-metre stages fr~m 0 up to 8 metres, and at each of these positions an observation was made of the effects of directing the normal towards and away from P respectively. The points at which no difference in the sparking was observed in the two positions of the normal are marked 0 in the table below. Those in which the sparking was least,

TABLE I.

	0		1		2		3		4		5		6		7		8
100	+	+	0	–	–	–	–	–	0	0	0	0	0	+	+	+	+
150	+	0	–	–	–	–	0	0	0	0	0	+	+	+	+	+	0
200	0	–	–	–	–	–	0	+	+	+	+	+	0	0	0	0	0
250	0	–	–	–	–	0	0	+	+	+	+	0	0	0	0	0	0
300	–	–	–	–	0	+	+	+	+	+	0	0	0	0	–	–	–
350	–	–	0	+	+	+	+	+	+	0	0	0	–	–	–	–	–
400	–	–	0	+	+	+	+	0	0	0	0	–	–	–	–	–	–
450	–	0	+	+	+	+	+	0	0	0	–	–	–	–	–	–	0
500	–	0	+	+	+	+	0	–	–	–	–	–	0	0	0	0	+
550	0	+	+	+	+	0	0	–	–	–	–	–	0	0	0	0	+
600	+	+	+	+	0	0	–	–	–	–	–	0	0	+	+	+	+

showing the existence of interference, when the normal was directed towards P are marked +, and those in which the sparking was least when the normal was directed away from P are marked –. The experiments were repeated with different lengths of wire *m n*, varying by steps of half a metre from 1 metre up to 6 metres. The first horizontal line in the table gives the distance, in metres, of the centre of the secondary circuit from the null point, while the first vertical line gives the lengths of the wire *m n*, also in metres.

An inspection of this table shows, in the first place, that the changes of sign take place at longer intervals than 2·8 metres; and, in the second place, that the change of phase is more rapid in the neighbourhood of the origin than at a distance from it. As a variation in the velocity of propagation is very unlikely, this is probably due to the fact indicated by theory that the electrostatic E.M.F., which is more powerful than the electro-dynamic E.M.F. in the neighbourhood of the primary oscillation, has a greater velocity of propagation than the latter.

In order to obtain a definite proof of the existence of similar phenomena at greater distances, Dr. Hertz continued the observations, in the case of three of the lengths $m\,n$, up to a distance of 12 metres, and the result is given in the table below:—

TABLE II.

	0	1	2	3	4	5	6	7	8	9	10	11	12
100	+	0	−	−	0	0	0	+	+	+	+	+	0
250	0	−	−	0	+	+	0	0	0	0	−	−	−
400	−	0	+	+	0	0	−	−	−	−	0	0	0

If we make the assumption that at the greater distance it is only the E.M.F. of induction which produces any effect, the experiments would show that the interference of the waves excited by the E.M.F. of induction with the original waves in the wire changes its sign only at intervals of about 7 metres.

In order to investigate the E.M.F. of induction close to the primary oscillation, where the results are of special importance, Dr. Hertz made use of the interferences which were obtained when the secondary circuit was in the third principal position, and the air space was rotated through 90deg. from the base-line. The direction of the interference at the null point, which has already been considered, was taken as negative, the interference being considered positive when it was produced by the passage of waves on the side of C remote from P, which makes the signs correspond with those of the previous experiments. It must be borne in mind that the direction of the resultant E.M.F. at the null point is opposed to that of the E.M.F. of induction, and therefore the first table would have begun with a negative sign if the electrostatic E.M.F. could have been eliminated. The present experiments showed that up to a distance of 3 metres interference continued to occur, and always of the same sign as at the null point. It was unfortunately impossible to extend these observations to a greater distance than 4 metres, on account of the feebleness of the sparks, but the results obtained were sufficient to give distinct evidence of a finite velocity of propagation of the E.M.F. of induction. These observations, like the former ones, were repeated with various lengths of the wire mn in order to exhibit the variation in phase, and the results obtained are given in the table below, which shows

TABLE III.

	0	1	2	3	4
100	−	−	−	−	0
150	−	−	0	0	0
200	0	0	0	+	+
250	0	+	+	+	+
300	+	+	+	+	+
350	+	+	+	+	0
400	+	+	+	+	0
450	+	+	+	0	0
500	+	+	0	0	0
550	+	0	0	0	−
600	0	−	−	−	−

that as the distance increases, the phase of the interference changes in such a manner that a reversal of sign takes place at intervals of from 7 to 8 metres. This result is further confirmed by comparing the results of Table III. with the results for greater distances given in Table II., for in the former series the effect of the electrostatic E.M.F. is eliminated, owing to the special position of the secondary circuit, while in the former it becomes insensible at the greater distances owing to its rapid decrease with increasing distance. We should therefore expect the results given in the first table for distances beyond 4 metres to follow without a break the results given in Table III. for distances up to 4 metres. This was found to be the case, as is evident from inspection of Tables II. and III.

To show this more clearly, the signs of the interference of the waves, due to the electro-dynamic E.M.F., with the waves in the wire are collected together in Table IV., the first four columns of which are taken from Table III., and the remaining columns from Table II.

TABLE IV.

	0	1	2	3	4	5	6	7	8	9	10	11	12
100	–	–	–	–	0	0	0	+	+	+	+	+	0
250	0	+	+	+	+	+	0	0	0	0	–	–	–
400	+	+	+	+	0	0	–	–	–	–	0	0	0

From the results given in this table the author draws the following conclusions:—

1. The interference does not change its sign at intervals of 2·8 metres. The electro-dynamic actions are therefore not propagated with an infinite velocity.

2. The interference is not in the same phase at all points. Therefore the electro-dynamic actions are not propagated through air with the same velocity as electric waves in wires.

3. A gradual retardation of the waves in the wire has the effect of displacing a given phase of the interference towards the origin of the waves. The velocity of propagation through the air is therefore greater than through a wire.

4. The sign of the interference is reversed at intervals of 7·5 metres, and therefore in traversing this distance an electro-dynamic wave gains one length of the waves in the wire.

Thus, while the former travels 7·5 metres, the latter travels $7{\cdot}5 - 2{\cdot}8 = 4{\cdot}7$ metres, and therefore the ratio of the velocities is 75 : 47, which gives for the half wave-length of the electro-dynamic action $2{\cdot}8 \times 75/47 = 4{\cdot}5$ metres. Since this distance is traversed in 1·4 hundred-millionth of a second, the absolute velocity of propagation through the air must be 320,000 kilometres per second. This result can only be considered reliable as far as its order is concerned; but its true value can hardly exceed half as much again, or be less than two-thirds of this amount. In order to obtain a more accurate determination of the true value it will be necessary to determine the velocity of electric waves in wires with greater exactness.

It does not necessarily follow from the fact that in the immediate neighbourhood of the primary oscillation the interference changes its sign after an interval of 2·8 metres, that the velocity of propagation of the electrostatic action is infinite, for such a conclusion would rest upon a single change of sign, which might, moreover, be explained, independently of any change of phase, by a change in the sign of the amplitude of the resultant force at a certain distance from the primary oscillation. Quite independently, however, of any knowledge of the velocity of propagation of electrostatic actions, there exist definite proofs that the rates of propagation of electrostatic and electro-dynamic E.M.F.s are unequal.

In the first place, the total force does not vanish at any point on the base-line. Now, near the primary the electrostatic E.M.F. is the greater, while the electro-dynamic E.M.F. is the greater at greater distances. There must, therefore, be some point at which they are equal, and since they do not balance they must take different times to reach this point.

In the second place, the existence of points at which the direction of the resultant E.M.F. becomes indeterminate does not seem capable of explanation, except on the supposition that the electrostatic and electro-dynamic components perpendicular to each other are in appreciably different phases, and, therefore, do not compound into a rectilinear oscillation in a fixed direction. The fact that the two components of the resultant are propagated with different velocities is of considerable importance, in that it gives an independent proof that one of them at any rate must have a finite velocity of propagation.

Further researches of Dr. Hertz on electrical oscillations, of which accounts have been published, are to be found described in a Paper, "On Electro-Dynamic Waves in Air, and their Reflection," in Wiedemann's *Annalen*, Vol. XXXIV., p. 609. The author had been endeavouring to find a more striking and direct proof of the finite velocity of propagation of electro-dynamic waves than those which he had hitherto given; for, though these are quite sufficient to establish the fact, they can only be properly appreciated by one who has obtained a grasp of the results of the entire series of researches.

In many of the experiments which have been described, Dr. Hertz had noticed the appearance of sparks at points in the secondary conductor where it was clear from geometrical considerations that they could not be due to direct action, and it was observed that this occurred chiefly in the neighbourhood of solid obstacles. It was found, moreover, that in most positions of the secondary conductor the feeble sparks produced at a great distance from the primary became considerably stronger in the vicinity of a solid wall, but disappeared with considerable suddenness quite close to the wall. The most obvious explanation of these experiments was that the waves of inductive action were reflected from the wall and interfered with the direct waves, especially as it was found that the phenomena became more distinct when the circumstances were such as to favour reflection to the greatest possible extent. Dr. Hertz therefore determined upon a thorough investigation of the phenomena.

The experiments were made in the Physical Lecture Theatre, which is 15 metres in length, 14 metres in width, and 6 metres in height. Two rows of iron columns, running parallel to the sides of the room, would collectively act almost like a solid wall towards electro-dynamic action, so that the available width of the room was only 8·5 metres. All pendant gas-fittings were removed, and the room left empty, with the exception of wooden tables and forms, which would not exert any appreciable disturbing effect. The end wall, from which the waves were to be reflected, was of solid sandstone, with two doors in it, and the numerous gas pipes attached to it gave it, to a certain extent, the character of a conducting surface, and this was increased by fastening to it a sheet of zinc four metres high and two metres broad, connected by wires to the gas pipes and a neighbouring water pipe. Special care was taken to provide an escape for the electricity at the upper and lower extremities of the zinc plate, where a certain accumulation of electricity was to be expected.

The primary conductor was the same that was employed in the experiments described on page 448, Fig. 34, and was placed at a distance of 13 metres from the zinc plate, and, therefore, two metres from the wall at the other end of the room. The conducting wire was placed vertically, so that the E.M.F.s to be considered increased and diminished in a vertical direction. The centre of the primary conductor was 2·5 metres above the floor of the room, which left a clear space for the observations above the tables and benches. The point of intersection of the reflecting surface with the perpendicular from the centre of the primary conductor will be called "the point of incidence," and the experiments were limited to the neighbourhood of this point, as the investigation of waves striking the wall at a considerable angle would be complicated by the differences in their polarisation. The plane of vibration was therefore parallel to the reflecting surface, and the plane of the waves was perpendicular to it, and passed through the point of incidence.

The secondary conductor consisted of the circle of 35 centimetres radius, which has been already described. It was movable about an axis through its centre perpendicular to its plane, and the axis itself was movable in a horizontal plane about a vertical axis. In most of the experiments the secondary conductor was held in the hand by its insulating wooden

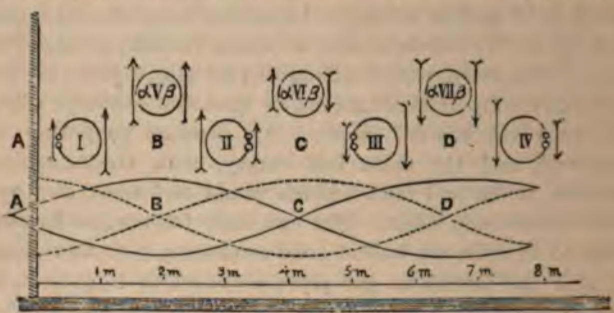

Fig. 35.

support, as this was the most convenient way of bringing it into the various positions required. The results of these experiments, however, had to be checked by observations made with the observer at a greater distance from the secondary, as the neighbourhood of his body exerted a slight influence upon the phenomena. The sparks were distinct enough to be observed at a distance of several metres when the room was darkened, but when the room remained light they were practically invisible even when the observer was quite close to the secondary.

When the centre of the secondary was placed in the line of incidence, and with its plane in the plane of vibration, and the air space was turned first towards the reflecting wall and then away from it, a considerable difference was generally observed in the strength of the sparks in the two positions. At a distance of about 0·8 metre from the wall the sparks were much stronger when the air space was directed towards the wall, and its length could be adjusted so that, while there was a steady stream of sparks when in this position, they disappeared entirely when the air space was directed away from the wall. These phenomena were reversed at a distance of 3 metres, and recurred, as in the first case, at a distance of 5·5 metres. At a distance of 8 metres the sparks were stronger when the air space was turned away from the wall, as at the distance of 3 metres, but the difference was not so well marked. When the distance was increased beyond 8 metres no further reversal took place, owing to the increase in the direct effect of the primary oscillation and the complicated distribution of the E.M.F. in its neighbourhood.

The positions I., II., III. and IV. (Fig. 35) of the secondary circle are those in which the sparks were strongest, the distance from the wall being shown by the horizontal scale at the foot. When the secondary circle was in the positions V., VI., and VII., the sparks were equally strong in both positions of the air space, and quite close to the wall the difference between the sparking in the two positions again diminished. Therefore the points A, B, C, D in the diagram may in a certain sense be regarded as nodes. The distance between two of these points must not, however, be taken as the half wave-length, for if all the electrical motions changed their directions on passing through one of these points the phenomena observed in the secondary circuit would be repeated without variation, since the direction of oscillation in the air space is indifferent.

The conclusion to be drawn from the experiments is that in passing any one of these points part of the action is reversed, while another part is not. The experimental results, however, warrant the assumption that twice the distance between two of these points is equal to the half wave-length, and when this assumption is made the phenomena can be fully explained.

For suppose a wave of E.M.F., with oscillations in a vertical direction, to impinge upon the wall, and to be reflected with only slightly diminished intensity, thus giving rise to stationary waves. If the wall were a perfect conductor, a node would necessarily be formed in its surface, for at the boundary and in the interior of a perfect conductor the E.M.F. must be infinitely small. The wall cannot, however, be considered as a perfect conductor, for it was not metallic throughout, and the portion which was metallic was not of any great extent. The E.M.F. would therefore have a finite value at its surface, and would be in the direction of the impinging waves. The node, which in the case of perfect conductivity would occur at the surface of the wall, would, therefore, actually be situated a little behind it, as shown at A in the diagram. If, then, twice the distance A B—that is to say, the distance A C—is half the wave-length, the steady waves will be as represented by the continuous lines in Fig. 35. The E.M.F.s acting on each side of the circles, in the positions I., II., III., and IV., will, therefore, at a given moment be represented in magnitude and direction by the arrows on each side of them in the diagram. If, therefore, in the neighbourhood of a node, the air space is turned towards the node, the strongest E.M.F. in the circle will act under more favourable conditions against a weaker one under less favourable conditions. If, however, the air space is turned away from the node, the stronger E.M.F. acts under less favourable conditions against a weaker one under more favourable conditions. In the latter case the resultant action must be less than in the former, whichever of the two E.M.F.s has the greater effect, which explains the change of sign of the phenomenon at each quarter wave-length.

This explanation is further confirmed by the consideration that, if it is the true one, the change of sign at the points B and D must take place in quite a different manner from that of the point C. The E.M.F.s acting on the secondary circle, in the positions V., VI., and VII., are shown by the corresponding arrows, and it is clear that in the positions B and D, if the air space is turned from one side to the other, the vibration will change its direction round the circle, and therefore the sparking must, during the rotation, vanish either once or an uneven number of times. In the position C, however, the direction of vibration remains unaltered, and therefore the sparks must disappear an even number of times, or not at all.

The experiments showed that at B and D the sparking diminished as the air space receded from a, vanished at the highest point, and again attained its original value at the point β. At C, on the other hand, the sparking continued throughout the rotation, being a little stronger at the highest and lowest points. If, then, there is any change of sign in the position C, it must occur with very much smaller displacements than in the other positions, so that in any case there is a distinction such as is required between this and the other two cases.

Another very direct proof of the truth of Dr. Hertz's representation of the nature of the waves was obtained. If the secondary circle lies in the plane of the waves instead of in the plane of vibration, the E.M.F. must be equal at all points of the circle, and for a given position of the air space the sparking must be directly proportional to its intensity. When the experiment was made, it was found, as expected, that at all distances the sparking vanished at the highest and lowest points of the circle, and attained a maximum value at the points in the horizontal plane through the point of incidence.

The air space was then placed at such a point and close to the wall, and was then moved slowly away from the wall, when it was found that, while there was no sparking quite close to the metal plate, it began at a very small distance from it, rapidly increased, reached a maximum at the point B, and then diminished again. At C the sparking again became excessively feeble and increased as the circle was moved still further away. The sparking continued steadily to increase after this, as the motion of the circle was continued in the same direction, owing as before, to the direct action of the primary oscillation.

The curves shown by the continuous lines in Fig. 35 were obtained from the results of these experiments, the ordinates representing the intensity of the sparks at the distances represented by the corresponding abscissæ.

The existence in the electrical waves of nodes at A and C, and of loops at B and D, is fully established by the experiments which have been described; but in another sense the points B and D may be regarded as nodes, for they are the nodal points of a stationary wave of magnetic induction which, according to theory, accompanies the electrical wave and lags a quarter wave-length behind it.

This can easily be shown to follow from the experiments, for when the secondary circle is placed in the plane of vibration with the air space at its highest point, there will be no sparking if the E.M.F. is uniform throughout the space occupied by the secondary. This can only take place if the E.M.F. varies from point to point of the circle, and if its integral round the circle differs from zero. This integral is proportional to the number of magnetic lines of force passing backwards and forwards across the circle, and the intensity of the sparks may be considered as giving a measure of the magnetic induction, which is perpendicular to the plane of the circle. Now, in this position vigorous sparking was observed close to the wall, diminishing rapidly to zero as the point B was approached, then increasing to a maximum at C, falling to a well-marked minimum at D, and finally increasing continuously as the secondary approached still nearer to the primary. If the intensities of these sparks are taken as ordinates, positive and negative, and the distances from the wall as abscissæ, the curve shown by the dotted lines in Fig. 35 is obtained, which therefore represents the magnetic waves.

The phenomena observed in the first series of experiments described in this paper may therefore be regarded as due to the resultant electric and magnetic actions. The former changes sign at A and C, the latter at B and D, so that at each of these points one part of the action changes sign, while the other does not, and therefore the resultant action which is their product must change sign at each of these points, as was found to be the case.

When the secondary circle was in the plane of vibration the sparking in the vicinity of the wall was observed to be a maximum on the side towards the wall and a minimum at the opposite side, and as the circle was turned from one position to the other there was found to be no point at which the sparks disappeared. As the distance from the wall was increased, the sparks on the remote side gradually became weaker, and vanished at a distance of 1·08 metre from the wall. When the circle was carried further in the same direction the sparks appeared again on the side remote from the wall, but were always weaker than on the side next to it; the sparking, however, no longer passed from a maximum to a minimum merely, but vanished during the rotation once in the upper and once in the lower half of the circle. The two null points gradually receded from their original coincident positions, until at the point B they occurred at the highest and lowest points of the circle. As the circle was moved further in the same direction the null points passed over to the side next to the wall, and approached each other again, until, when the centre was at a distance of 2·35 metres from the wall, the two null points were again coincident. B must be exactly half-way between this point and the similar point previously observed, which gives 1·72 metre as the distance of B from the wall—a result which agrees, within a few centimetres, with that obtained by direct observation. Moving further in the direction of C, the sparking at different points of the circle became more nearly equal, until at C it was exactly so. In this position there was no null point, and as the distance was further increased the phenomena recurred in the same order as before.

Dr. Hertz found that the position of C could be determined within a few centimetres, the determinations of its distance from the wall varying from 4·10 to 4·15 metres; he gives its most probable value as 4·12 metres. The point B could not be observed with any exactness, the direct determinations varying from 6 to 7·5 metres as its distance from the wall. It could, however, be determined indirectly, for the distance between B and C being found to be 2·4 metres, taking this as the true value, A must have been 0·68 metre behind the surface of the wall, and 6·52 metres in front of it. The half wave-length would be 4·8 metres, and by an indirect method it was found to be 4·5 metres, so that the two results agree fairly well. Taking the mean of these as the true value, and the velocity of light as the velocity of propagation, gives as the vibration period of the apparatus 1·55 hundred-millionth of a second, instead of 1·4 hundred-millionth, which was the theoretically calculated value.

A second series of experiments was made with a smaller apparatus, and though the measurements could not be made with as much exactness as those already described, the results showed clearly that the position of the nodes depends only on the dimensions of the conductors, and not on the material of the wall.

Dr. Hertz states that after some practice he succeeded in obtaining indications of reflection from each of the walls. He was also able to obtain distinct evidence of reflection from one of the iron columns in the room, and of the existence of electro-dynamic shadows on the side of the column remote from the primary.

In the preceding experiments the secondary conductor was always placed between the wall and the primary conductor—that is to say, in a space in which the direct and reflected rays were travelling in opposite directions, and gave rise to stationary waves by their interference.

He next placed the primary conductor between the wall and the secondary, so that the latter was in a space in which the direct and reflected waves were travelling in the same direction. This would necessarily give rise to a resultant wave, the intensity of which would depend on the difference in phase of the two interfering waves. In order to obtain distinct results it was necessary that the two waves should be of approximately equal intensities, and therefore the distance of the primary from the wall had to be small in comparison with the extent of the latter, and also in comparison with its distance from the secondary.

To fulfil these conditions the secondary was placed at a distance of 14 metres from the reflecting wall, and, therefore, about 1 metre from the opposite one, with its plane in the plane of vibration, and its air space directed towards the nearest wall, in order to make the conditions as favourable as possible for the production of sparks. The primary was placed parallel to its former position, and at a perpendicular distance of about 30 centimetres from the centre of the reflecting metallic plate. The sparks observed in the secondary were then very feeble, and the air space was increased until they disappeared. The primary conductor was then gradually moved away from the wall, when isolated sparks were soon observed in the secondary, passing into a continuous stream when the primary was between 1·5 and 2 metres from the wall—that is, at the point B. This might have been supposed to be due to the decrease in the distance between the two conductors, except that as the primary conductor was moved still further from the wall the sparking again diminished, and disappeared when the primary was at the point C. After passing this point the sparking continually increased as the primary approached nearer to the secondary. These experiments were found to be easy to repeat with smaller apparatus, and the results obtained confirmed the former conclusion—that the position of the nodes depends only on the dimensions of the conductor, and not on the material of the reflecting wall.

Dr. Hertz points out that these phenomena are exactly analogous to the acoustical experiment of approaching a vibrating tuning-fork to a wall, when the sound is weakened in certain positions and strengthened in others, and also to the optical phenomena illustrated in Lloyd's form of Fresnel's mirror experiments; and as these are accepted as arguments tending to prove that sound and light are due to vibration, his investigations give a strong support to the theory that the propagation of electromagnetic induction also takes place by means of waves. They therefore afford a confirmation of the Faraday-Maxwell theory of electrical action. He points out, however, that Maxwell's, in common with other electrical theories, leads to the conclusion that electricity travels through wires with the velocity of light —a conclusion which his experiments show to be untrue. He states that he intends to make this contradiction between theory and experiment the subject of further investigation.

§ 12. Further Researches on Electro-Magnetic Radiation.— When once the conception had arisen that the result of electrical oscillations in a conductor is to propagate out into surrounding space radiations which are in all respects of the same nature as *light* except in that they cannot affect the eye, it became evident that a new experimental field had been opened, and one in which it would be possible to produce the electro-magnetic analogues of many familiar optical phenomena.

The reflection, refraction, dispersion, and polarisation of light waves are well-known optical phenomena. We can perform analogous experiments with rays of dark heat which differ only from light rays in having a greater wave-length, and in being thereby unable to affect the optic nerve. In performing these experiments with dark heat or non-luminous radiation we have to make use of the thermopile as a *perceiver* of the ray. The electro-magnetic radiation scattered from a conductor in which electric oscillations are set up differs again from light and dark heat in having a still larger wave-length. In performing experiments with electro-magnetic radiation we have seen that Dr. Hertz's invention of the electro-magnetic resonator puts us in possession of an apparatus which is the exact equivalent of a thermopile, or the human eye, as a ray localiser. The expectation that the electro-magnetic theory would receive confirmation by the realisation of many familiar optical experiments with the substitution of electro-magnetic radiation for light has not been disappointed.

In these recent electro-magneto-optic experiments of Dr. Hertz, the source of radiation is a divided metallic cylinder about one inch in diameter and twelve inches long. This is divided in halves, and the two parts separated by a small distance. They are respectively attached to the ends of the secondary coil of a small induction coil. When the coil is put in action, electrical oscillations are set up in these cylinders which result in the outward propagation of ethereal undulations of about two feet in wave-length and having a frequency of about five hundred million a second.

In order to *see* these waves, Dr. Hertz employs a resonator consisting of a metallic circuit having a small spark interval. With these simple appliances he has been able to show the reflection of the electro-magnetic waves from plane surfaces, and the concentration of radiation by parabolic mirrors of sheet zinc, repeating in fact the old experiment of the conjugate mirrors. The radiation from this source could, he found, be gathered up by one parabolic mirror, reflected to a second and concentrated again to a focus. Another achievement was the refraction of the rays by a great prism of pitch. Placed in the path of an electro-magnetic ray, he found that this pitch prism refracted it through an angle of 22deg., and that the material had a refractive index of 1·7 for these long waves. Again, it was found that metallic sheets were opaque to this radiation, but that it passed through such non-conductors as dry wood, and that a laboratory door, although opaque to light, is transparent to this ultra-ultra red or electro-magnetic radiation.

Ingenuity will not be wanting to complete in time all the necessary electro-magnetic analogues, and to build up an impregnable body of proof that will indicate that electro-magnetic radiation and what we commonly call *light* are one in essential nature, although differing in degree. These experiments are akin to the acoustic ones in which air waves, too short to be audible, are generated; and in place of the ear, now useless, a sensitive flame is employed to find or indicate the waves, and experiments analogous to well-known defraction experiments in light can be performed.

It is a necessary corollary of Maxwell's electro-magnetic theory of light that good conducting bodies should be opaque and good insulators transparent. As a matter of fact, for disturbances of the period of light many good insulators, such as ebonite, are opaque, even in very thin sheets, and conversely gold, silver, and platinum are semi-transparent when in very thin sheets. It must be borne in mind, however, that the frequency of light oscillations falls between 400 and 700 million-million oscillations per second, or are of the order of 5×10^{14}.

We cannot by any of Hertz's methods produce electrical oscillations so rapid as this. Hence, since conductivity and insulating power of materials have generally been determined with reference either to steady currents or to moderately great oscillations, we cannot institute a comparison between these qualities as possessed by any given substance and opacity or transparency for the much greater frequency of luminous electro-magnetic waves. It has been shown that ebonite is very transparent to long waves of dark heat,* and hence there is no difficulty in understanding that it is transparent to the longer waves produced by electrical oscillations set up in moderately small conductors, whilst it is opaque to the very short ones of light. Also the transparency of thin metallic sheets to light is an indication of imperfect conductibility. We have seen that an infinitely perfect conductor is a perfect magnetic screen, and accordingly we should expect that the more perfect the conductivity of a metal the greater would be its opacity even in very thin films. It is well known that cooling copper increases its conductivity. Wroblewski has shown (*Comptes Rendus*, Vol. CI., July, 1885, p. 160) that by cooling copper to $-200°$C., or to the temperature of the solidification of nitrogen, its conductivity is increased to about nine times its value at 0°C. It would be interesting to know if the opacity of a very thin film of copper is increased by a similar cooling to any perceptible extent. With respect to electrolytes some

* See note on "The Index of Refraction of Ebonite," by Profs. Ayrton and Perry, *Proceedings of the Physical Society, London*, Vol. IV., p. 345.

interesting experiments have been made by Prof. J. J. Thomson.* In these experiments electrical oscillations of about 10^8 per second in frequency were established in a primary circuit by means of an induction coil. These alternating currents were caused to induce secondary oscillations in a neighbouring parallel circuit of appropriate size. The secondary circuit oscillations were rendered visible by minute sparks at a break in that circuit. The interposition of a thin sheet of tinfoil or of the thinnest sheet of Dutch metal or gold-leaf supported on glass at once stopped completely the secondary sparks. This is a very interesting confirmation of the theory of magnetic screening laid down on p. 209 of Chap. IV. We have seen that for moderately rapid alternations the conductivity of tinfoil is not sufficient to make it opaque to electro-magnetic radiations, but for disturbances of a frequency equal to about 10^8 the tinfoil affords a perfect screening, or, in other words, is opaque.

With regard to the gold-leaf, Prof. Thomson remarks that he has not been able to get any leaf thin enough to be transparent to oscillations of this rate. On inserting a sheet of ebonite between the primary and secondary circuit, it was found to produce no effect on the sparking, indicating that ebonite, although opaque to ordinary light, is transparent to ether disturbances of the rate here employed. A thin layer of transparent electrolyte was then used as a screen, and it was found that whilst a very thin layer produced little or no effect, a depth of three to four millimetres of dilute sulphuric acid was sufficient to stop the sparking. Experiment showed that the conductivity of various electrolytic solutions was about the same for currents reversed 120 times a second as for currents reversed 100 million times a second. As, however, these electrolytes are transparent, they must, according to the electromagnetic theory, be insulators for currents reversed about 10^{15} times per second, and the molecular processes on which electrolytic conduction depends must occupy a time between one-hundred millionth and one-thousand billionth of a second.

* "On the Resistance of Electrolytes to the Passage of Rapidly Alternating Currents," *Proceedings of the Royal Society, London.* Vol. XLV., No. 276, 1889.

Space will not permit further reference to this exceedingly promising department of future research more than to say that if the electro-magnetic theory of light is true it will be able to furnish an electrical explanation, not only of the simpler optical phenomena, but of such complex phenomena as those embraced in the sciences of spectroscopy and photography.

§ 13. **Propagation of Electro-Magnetic Energy.**—In our exposition of the various electro-magnetic phenomena we have directed attention to the facts which make it evident that even in the simple phenomenon of the propagation of an electric current in a wire we must divest ourselves of the idea that the so-called flow of current is analogous to the movement of a material fluid in a pipe. It is true that there are effects in the case of the electric current which correspond to the inertia and resistance effects in the water flow; but the progress of knowledge has indicated that what we are in the habit of calling the electric current is as much outside the wire as in it, and that we must release ourselves from the trammels of any ideas which cause us to concentrate attention exclusively or mainly on the action *in* the conductor. We are indebted to Prof. Poynting for an enlargement of our views on the nature of electric current propagation, and in two valuable memoirs these matters have been discussed by him.*

He says:—A space containing electric currents may be regarded as a field where energy is transformed at certain points into the electric and magnetic kind by means of batteries, dynamos, thermopiles, &c., and in other parts of the field this energy is being again transformed into heat, work done by electro-magnetic forces, or any other form yielded by currents. Formerly a current was regarded as something travelling in the conductor, and the energy which appeared at any part of the circuit was supposed to be conveyed thither through the conductor by the current. But the existence of induced currents

* "On the Transfer of Energy in the Electro-Magnetic Field," by Prof. J. H. Poynting, *Philosophical Transactions of the Royal Society*, 1884, Part II., Vol. CLXXV., p. 343. Also "On the Connection between Electric Current and the Electric and Magnetic Inductions in the Surrounding Field," by Prof. J. H. Poynting, *Philosophical Transactions of the Royal Society*, 1885, Part II., Vol. CLXXVI., p. 277.

and electro-magnetic actions has led us to look on the medium surrounding the conductor as playing a very important part in the development of the phenomena. If we believe in the continuity of the motion of energy, we are forced to conclude that the surrounding medium is capable of containing energy, and that it is capable of being transferred from point to point. We are thus led to consider the problem—how does the energy about an electric current pass from point to point, by what paths does it travel, and according to what laws? Let us put a specific case. Suppose a dynamo at one spot generates an electric current, which is made to operate an electric motor at a distant place. We have here in the first place an absorption of energy from the prime motor into the dynamo. We find the whole space between and around the conducting wires magnetised, and the seat of electro-magnetic energy. We have further a retransformation of energy in the motor. The question which presents itself for solution is to decide how the energy taken up by the dynamo is transmitted to the motor, by what paths it travels, and according to what laws. Briefly stated, the tendency of recent views is that this energy is conveyed through the electro-magnetic medium or ether, and that the function of the wire is to localise the direction or concentrate the flow in a particular path, and to afford a *sink* or place in which energy can be dissipated. A consideration of the whole phenomena has enabled Prof. Poynting to formulate an important law, as follows :—*At any point in the magnetic field of conductors conveying currents the energy moves perpendicularly to the plane containing the lines of electric force and the lines of magnetic force, and the amount crossing a unit of area of this plane per second is equal to the product of the intensities of the two forces multiplied by the sine of the angle between them and divided by* 4π. If E denote the electric force or force on a very small body charged with a unit of positive electricity, and H denote the magnetic force or force on a small free unit North Pole, and if at any point in the electro-magnetic field these forces are inclined at an angle θ, then there is a flow of energy e at this point in a direction perpendicular to the planes of E and H, and equal per second to the value of

$$\frac{E\,H\sin\theta}{4\pi}.$$

The full proof of this law is given in the first of the Papers mentioned above.

Prof. Poynting has here introduced the important notion of a flow of energy. We may remark in passing that this notion does no violence to previous notions of energy. Energy, like matter, is *conserved*—that is, it is unalterable in total amount; and if in any circumscribed space some form of energy makes its appearance, then we know that either an equal quantity must have passed into that space from outside, or that an equivalent quantity of some other form already in the enclosure must have been transformed. If energy disappears at one point and reappears at an adjacent point in equal amount, we can with perfect propriety speak of it as having been transferred from one point to the other, although we are unable to identify the respective portions of it as we can in the case of the movement of matter. Applying this view to the simple phenomena of a battery producing heat in a conducting wire, the notion to be grasped is that the potential energy of the chemical combinations in the battery causes energy to be radiated out along certain lines, the means of conveyance being the electro-magnetic medium; this energy flows into the wire at all points, and is there re-transformed into heat or light. A simple illustration of Poynting's law is to consider the case of a section of a straight conductor traversed, as we usually say, by a current. Let the conductor be a right cylinder, or round wire, of length l, radius r, and let E be the electric force at any point in the wire, and H the magnetic force at the surface; also let V be the potential difference between the ends, C the steady current, and R the total ohmic resistance. Consider the energy flowing in on this section of the wire through its surface. It is equal per second to the area of the surface multiplied by $\frac{EH}{4\pi}$, or to $\frac{2\pi r l E H}{4\pi}$.

Now $2\pi r H$ is the line integral of the magnetic force taken round the wire following the circular surface, and this, as previously shown (p. 354) is equal to $4\pi C$. Also we have the potential difference at the ends of the cylinder equal to the line integral of the electric force, or to $l E$. Since, then, $2\pi r H = 4\pi C$ and $E l = V$, we get by substitution in the value of the energy sent per second into the section of the wire, viz., $\frac{2\pi r l \text{H E}}{4\pi}$, the equivalent C V. But by Ohm's law C R = V; hence the energy absorbed per second by the conductor is C^2 R, and we know by Joule's law that this is the measure of the energy dissipated per second in the wire as heat. We see, then, that the energy dissipated in each section of the conductor is absorbed into it from the dielectric, and the rate of this supply can be calculated by Poynting's law for each element of the surface. None of the energy of a current travels along the wire, but it enters into it from the surrounding non-conductor, and as soon as it enters it begins to be transformed into heat, the amount crossing successive layers of the wire decreasing till by the time the centre is reached where there is no magnetic force it has all been transformed into heat. In the original Paper another simple case treated is that of a condenser discharged by a wire. In this case, before the discharge, we know that the energy resides in the dielectric between the plates. If the plates are connected by an external wire, according to these views the energy is transferred outwards, along the electrostatic equipotential surfaces, and moves *on* to the wire and is there converted into heat. According to this hypothesis, we must suppose the lines of electrostatic induction running from plate to plate to move outwards as the dielectric strain lessens and whilst still keeping their ends on the plates to finally converge in on the wire and be there broken up and their energy dissipated as heat. At the same time the wire acquires transient magnetic qualities. This means that some part of the energy of the expanding lines of electrostatic induction is converted into magnetic energy. The magnetic energy is contained in ring-shaped tubes of magnetic force which expand out from between the plates and then contract in upon some other part of the circuit.

The whole history of the discharge may be divided into three parts. First, a time when the energy associated with the system is nearly all electrostatic and is represented by the energy of the lines or tubes of electrostatic induction running from plate to plate; second, a period when the discharge is at its maximum, when the energy exists partly as energy associated with lines of electrostatic induction expanding outwards, and partly in the form of closed rings or tubes of magnetic force expanding and then contracting back on the wire; and then, lastly, a period when nearly all the energy has been absorbed or buried in the wire, and has there been dissipated in the form of heat, which is radiated out again as energy of dark or luminous radiation. The function of the discharging wire is to localise the place of dissipation, and also to localise the place where the magnetic field shall be most intense, and all that observation is able to tell us about a conductor which is conveying that which we call an electric current is that it is a place where heat is being generated, and near which there is a magnetic field. These conceptions lead us to fresh views of very familiar phenomena. Suppose we are sending a current of electricity through a submarine cable by a battery, say, with zinc to earth, and suppose the sheath is everywhere at zero potential, then the wire will be everywhere at a higher potential than the sheath, and the level surfaces will pass through the insulating material to the points where they cut the wire. The energy which maintains the current and which works the needle at the further end travels through the insulating material, the core serving as a means to allow the energy to get into motion or to be continually propagated. This energy sucked up by the core is, however, transformed into heat and radiated again as dark heat. If we adopt the electro-magnetic theory of light, it moves out again still as electro-magnetic energy, but in a different form, with a definite velocity and intermittent in type. We have then in the case of the electric light this curious result—that energy moves in upon the arc or filament from the surrounding medium, there to be converted into a form in which it is sent out again, and through which the same in kind is able to affect our senses.

In the case of an arc or glow-lamp worked by an alternating current, we have still further the result that the energy which moves in on the carbon is returned again, with no other change than that of a shortened wave-length, and the carbon filament performs the same kind of change on the electro-magnetic radiation as is performed when we heat a bit of platinum foil to vivid incandescence in a focus of dark heat. A current through a seat of electromotive force is therefore a place of divergence of energy from the conducting circuit into the medium, and this energy travels away and is converged and transformed by the rest of the circuit. From this aspect the function of the copper conducting wire fades into insignificance in interest in comparison with the function of the dielectric, or rather of the ether contained in the dielectric. When we see an electric tramcar, or motor, or lamp worked from a distant dynamo, these notions invite us to consider the whole of that energy, even if it be thousands of horse-power per hour, as conveyed through the ether or magnetic medium, and the conductor as a kind of exhaust valve, which permits energy to be continually supplied to the dielectric.

Consider, for instance, the simple case of an alternating current-dynamo connected to an incandescence lamp by conducting leads. We have in this case a closed conducting loop, consisting partly of the armature wire, partly of the leads, and lastly of the lamp filament. The action of the dynamo when at work consists in alternately inserting into and withdrawing a bundle of lines of magnetic induction from a portion of this enclosed area or loop. The insertion of these lines of force causes an electro-magnetic disturbance which travels away through the enclosed dielectric in the form of some strain or displacement in its most generalised sense. In reaching the surface of the enclosing conductor this wave begins to soak into it, the electro-magnetic energy at the same time dissipating itself in it in the form of heat. By a suitable arrangement of the resistances and surfaces of various portions of the circuit, we are able to localise the principal place of transformation, and to control its rate so as to compel this transformation of energy to take place at a certain rate in a limited portion of the conductor. Energy is then sent out thence again in a radiant form, partly in the form of ether waves capable of exciting the retina of the eye, but very largely in the form of dark heat. The ether, or electro-magnetic medium, is, therefore, the vehicle by which the energy is carried to the lamp, and conveyed away from it in an altered form, and whatever be the translating device employed, the ether is the seat of the hidden operations, which are really the fundamental ones, and the visible apparatus only the contrivances by which the nature of the energy transformation is determined and its place defined.

These views are the outcome of that half-century of scientific thought which dates from the period of Faraday's conception of an electro-magnetic medium. We can without hesitation predict that the ideas which have thus guided to so much discovery are destined yet to be the clue to conduct to further revelations of the nature of the unseen mechanism which lies behind the apparent operations on, or behaviour of, the bodies in the electro-magnetic field and of which these actions are the result. If the nature of that machinery, as yet shrouded in darkness, should prove to be capable of more entire elucidation by some vortex motion theory, not only explaining the structure of matter itself, but the inmost actions of the electro-magnetic medium, then the sciences of electricity, magnetism, optics, electro-chemistry, and radiation will become only departments of one embracing science of hydrodynamics, whilst we shall yet find ourselves in the presence of still more surpassingly complex puzzles as to the nature of Energy itself and its relation to the physical structure of the electro-magnetic medium.

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